Solving a Quartic Equation

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1 A Reduction

By substituting

$$y = x + \frac{a_{n-1}}{na_n},$$

the problem of solving a polynomial equation of degree n

$$\sum_{k=0}^{n} a_k x^k = 0$$

can be reduced to the problem of solving another polynomial equation

$$y^n + b_{n-2}y^{n-2} + b_{n-3}y^{n-3} + \cdots + b_2 y^2 + b_1 y + b_0 = 0$$

where the leading coefficient is one and the second term is missing. That is why below we only focus on polynomial equations of this form.

2 Solve $x^3 + px = q$

Compared to $(s+t)^3 - 3st(s+t) = s^3 + t^3$ we have

$$\begin{cases} st & = -\frac{p}{3} \\ s^3 + t^3 & = q \end{cases},$$

and hence

$$s = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \qquad t = \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},$$

and

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},$$

where we find the cubic root such that $st = -p/3$ holds.

3 Solve $x^4 + px^2 + qx + r = 0$

Completing the square, we have

$$\left(x^2 + \frac{p}{2}\right)^2 = -qx - r + \frac{p^2}{4}.$$

Insert u into the LHS to obtain

$$\left(x^2 + \frac{p}{2} + u\right)^2 = 2ux^2 - qx + pu + u^2 - r + \frac{p^2}{4}. \tag{1}$$

Now find the u such that the RHS becomes a perfect square, that is

$$q^2 - 4(2u)\left(pu + u^2 - r + \frac{p^2}{4}\right) = 0,$$

or

$$8u^3 + 8pu^2 + (2p^2 - 8r)u - q^2 = 0.$$

Once such u is found, due to the first two terms of the RHS of Eq. (1), we know that

$$\left(x^2 + \frac{p}{2} + u\right)^2 = \left(\sqrt{2u}x - \frac{q}{2\sqrt{2u}}\right)^2,$$

and hence

$$\left(x^2 + \frac{p}{2} + u\right) = \pm\left(\sqrt{2u}x - \frac{q}{2\sqrt{2u}}\right).$$

Solve these qudratic functions to obtain the roots of our quartic function.