# Error Estimate for the Newton-Raphson Method

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**Abstract**

The error of Newton's method is estimated. First a special case is shown to demonstrate how powerful it is, and then more detail analysis for general cases is given.

## 1 An Example

Define

$$\begin{aligned} x_0 &= 2, \\ x_{n+1} &= \frac{x_n + \frac{3}{x_n}}{2}. \end{aligned}$$

This sequence will finally converge to $\sqrt{3}$. Define the error sequence $e_n = x_n - \sqrt{3}$, and consider $e_{n+1}$. Note that

$$\begin{aligned} e_{n+1} &= \frac{x_n + \frac{3}{x_n}}{2} - \sqrt{3} \\ &= \frac{1}{2x_n}\left(x_n^2 - 2\sqrt{3}x_n + 3\right) \\ &= \frac{e_n^2}{2x_n}. \end{aligned}$$

Thus we have

$$\begin{aligned} e_1 &= \frac{e_0^2}{2x_1} \\ e_2 &= \frac{1}{2x_2}\left(\frac{e_0^2}{2x_1}\right) = \frac{e_0^4}{2^{1+2}x_2x_1^2} \\ e_3 &= \frac{1}{2x_3}\left(\frac{e_0^4}{2^{1+2}x_2x_1^2}\right)^2 = \frac{e_0^8}{2^{1+2+4}x_3x_2^2x_1^4} \\ &\vdots \\ e_n &= \frac{e_0^{2^n}}{2^{2^n-1}x_nx_{n-1}^2x_{n-2}^4x_{n-3}^8\ldots x_1^{2(n-1)}} \leq 2\sqrt{3}\left(\frac{e_0}{2\sqrt{3}}\right)^{2^n}, \end{aligned}$$

where the last inequality holds because $x_k > \sqrt{3}$ for all $k = 1, 2, 3, \ldots$. Since $2\sqrt{3} \approx 3.464$, and $e_0 = 2 - \sqrt{3} \approx 0.268$, we have

$$\begin{aligned} e_n &\le 2\sqrt{3}\left(\frac{e_0}{2\sqrt{3}}\right)^{2^n} \approx (3.464)\left(\frac{0.268}{3.464}\right)^{2^n} \\ &\approx (3.464)(0.077)^{2^n} < 10^{-(2^n-1)}, \end{aligned}$$

which ensures a $(2^n - 1)$-decimal-place accuracy.

# 2 General Case

Let $c$ be a real root of a smooth function $f(x)$, that is $f(c) = 0$. Define

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ e_n &= x_n - c. \end{aligned}$$

Similar to the above analysis, we have

$$\begin{aligned} e_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - c \\ &= e_n - \frac{f(x_n)}{f'(x_n)} \\ &= \frac{1}{f'(x_n)}\left(e_n f'(x_n) - f(x_n)\right). \end{aligned} \tag{1}$$

By Taylor expansion and the fact that $f(c) = 0$, we can write

$$\begin{aligned} f(x_n) &= (x_n - c)f'(c) + (x_n - c)^2\frac{f''(c)}{2!} + (x_n - c)^3\frac{f'''(c)}{3!} + \cdots, \\ f'(x_n) &= f'(c) + (x_n - c)f''(c) + (x_n - c)^2\frac{f'''(c)}{2!} + (x_n - c)^3\frac{f''''(c)}{3!} + \cdots. \end{aligned}$$

Keep in mind all $(x_n - c)$ above can be replaced by $e_n$. Substitute the above sequences into Eq. (1) to obtain

$$\begin{aligned} e_{n+1} &= \frac{1}{f'(x_n)}\left(e_n\left(f'(c) + e_n f''(c) + e_n^2\frac{f'''(c)}{2!} + e_n^3\frac{f''''(c)}{3!} + \cdots\right)\right. \\ &\quad \left. - \left(e_n f'(c) + e_n^2\frac{f''(c)}{2!} + e_n^3\frac{f'''(c)}{3!} + \cdots\right)\right) \\ &= \frac{e_n^2}{f'(x_n)}\left(\left(1 - \frac{1}{2}\right)f''(c) + \left(\frac{1}{2!} - \frac{1}{3!}\right)e_n f'''(c) + \left(\frac{1}{3!} - \frac{1}{4!}\right)e_n^2 f''''(c) + \cdots\right). \end{aligned} \tag{2}$$

If $f(x)$ is a polynomial, there are only finitely many terms in the above sequence. As for general smooth functions, the remainder term of the Taylor expansion should be involved for more detail analysis. In fact, whether or not Newton's method converges to the root we want to find strongly depends on the behavior of this sequence.

Most literatures suggest the following approximation for the error of Newton's method

$$e_{n+1} \approx \frac{f''(x_n)}{2f'(x_n)} e_n^2. \tag{3}$$

To see how accurate this approximation is, we apply the Taylor expansion again

$$f''(x_n) \quad = \quad f''(c) + e_n f'''(c) + e_n^2 \frac{f''''(c)}{2!} + \cdots .$$

Multiply the right hand side by $e_n^2/(2f'(x_n))$ and subtract the resulting sequence by Eq. (2) to obtain

$$\frac{e_n^2}{f'(x_n)} \left( \left( \frac{1}{2} - \left( \frac{1}{2!} - \frac{1}{3!} \right) \right) e_n f'''(c) + \left( \frac{1}{2 \cdot 2!} \left( \frac{1}{3!} - \frac{1}{4!} \right) \right) e_n^2 f''''(c) + \cdots \right).$$

This is the difference between the "rough approximation" for the error of Newton's method given in Eq. (3), and the exact error given in Eq. (2). In most cases it is actually quite small.