Merton's Jump-Diffusion Model

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1 Stochastic Differential Equation for S_t

In Merton's jump-diffusion model for option pricing, the price process of an underlying asset S_t is assumed to follow the stochastic differential equation (sde)

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t + S_{t^-} \, d\left(\sum_{j=0}^{N(t)} (V_j - 1)\right),$$

where V_j are i.i.d. log-normal random variables. This SDE can be solved to obtain

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \prod_{j=0}^{N(t)} V_j.$$

Set $Y_j \equiv \log V_j \sim N(\alpha, \beta^2)$ and rewrite S_t to obtain

$$X_t \equiv \log \frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t + \sum_{j=0}^{N(t)} Y_j.$$

2 The Characteristic Function of X_t

Since W_t, $N(t)$, Y_j are mutually independent, the characteristic function of X_t is

$$\begin{aligned}
\mathbb{E}\left(e^{iuX_t}\right) &= \mathbb{E}\left(e^{iu\left(\mu - \frac{\sigma^2}{2}\right)t}\right) \mathbb{E}\left(e^{iu\sigma W_t}\right) \mathbb{E}\left(\exp\left\{iu \sum_{j=0}^{N(t)} Y_j\right\}\right) \\
&= e^{iu\left(\mu - \frac{\sigma^2}{2}\right)t} e^{-\frac{u^2\sigma^2}{2}t} \mathbb{E}\left(\exp\left\{iu \sum_{j=0}^{N(t)} Y_j\right\}\right),
\end{aligned}$$

where

$$
\begin{aligned}
\mathbb{E}\left(\exp\left\{iu\sum_{j=0}^{N(t)}Y_j\right\}\right) &= \mathbb{E}\left(e^{iuY_0+iuY_1+iuY_2+\cdots+iuY_{N(t)}}\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(e^{N(t)iuY_0}\middle| N(t)\right)\right) \\
&= \sum_{k=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^k}{k!}\left(\mathbb{E}\left(e^{iuY_0}\right)\right)^k. \qquad (1)
\end{aligned}
$$

Let Z be a standard normal random variable. Since $Y_0 \sim \alpha + \beta Z$, we have

$$
\mathbb{E}\left(e^{iuY_0}\right) = e^{iu\alpha}\mathbb{E}\left(e^{iu\beta Z}\right) = e^{iu\alpha - \frac{\beta^2 u^2}{2}}.
$$

Substitute this into Eq. (1) to obtain

$$
\begin{aligned}
\mathbb{E}\left(\exp\left\{iu\sum_{j=0}^{N(t)}Y_j\right\}\right) &= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{\left(\lambda t e^{iu\alpha-\frac{\beta^2u^2}{2}}\right)^k}{k!} \\
&= e^{\lambda t\left(e^{iu\alpha-\frac{\beta^2u^2}{2}}-1\right)},
\end{aligned}
$$

and hence the characteristic function

$$
\mathbb{E}\left(e^{iuX_t}\right) = e^{iu\left(\mu-\frac{\sigma^2}{2}\right)t-\frac{u^2\sigma^2}{2}t+\lambda t\left(e^{iu\alpha-\frac{\beta^2u^2}{2}}-1\right)}. \qquad (2)
$$

3 Cumulants

Substituting $u = -is$ into Eq. (2), we obtain the moment generating function of X_t

$$
\mathbb{E}\left(e^{sX_t}\right) = e^{s\left(\mu-\frac{\sigma^2}{2}\right)t+\frac{s^2\sigma^2}{2}t+\lambda t\left(e^{s\alpha+\frac{\beta^2s^2}{2}}-1\right)},
$$

and hence by definition the cumulant generating function

$$
\ln \mathbb{E}\left(e^{sX_t}\right) = s\left(\mu-\frac{\sigma^2}{2}\right)t+\frac{s^2\sigma^2}{2}t+\lambda t\left(e^{s\alpha+\frac{\beta^2s^2}{2}}-1\right),
$$

the power series expansion of which gives the cumulants. Here are the first few cumulants κ_n:

$$
\begin{aligned}
\kappa_1 &= \alpha\lambda t + t\left(\mu - \frac{\sigma^2}{2}\right), \\
\kappa_2 &= t\left(\sigma^2 + \lambda\left(\alpha^2 + \beta^2\right)\right), \\
\kappa_3 &= \lambda t\left(\alpha^3 + 3\alpha\beta^2\right), \\
\kappa_4 &= \lambda t\left(\alpha^4 + 6\alpha^2\beta^2 + 3\beta^4\right), \\
\kappa_5 &= \lambda t\left(\alpha^5 + 10\alpha^3\beta^2 + 15\alpha\beta^4\right), \\
\kappa_6 &= \lambda t\left(\alpha^6 + 15\alpha^4\beta^2 + 45\alpha^2\beta^4 + 15\beta^6\right), \\
\kappa_7 &= \lambda t\left(\alpha^7 + 21\alpha^5\beta^2 + 105\alpha^3\beta^4 + 105\alpha\beta^6\right), \\
\kappa_8 &= \lambda t\left(\alpha^8 + 28\alpha^6\beta^2 + 210\alpha^4\beta^4 + 420\alpha^2\beta^6 + 105\beta^8\right), \\
\kappa_9 &= \lambda t\left(\alpha^9 + 36\alpha^7\beta^2 + 378\alpha^5\beta^4 + 1260\alpha^3\beta^6 + 945\alpha\beta^8\right), \\
\kappa_{10} &= \lambda t\left(\alpha^{10} + 45\alpha^8\beta^2 + 630\alpha^6\beta^4 + 3150\alpha^4\beta^6 + 4725\alpha^2\beta^8 + 945\beta^{10}\right).
\end{aligned}
$$

In fact, for all $n \geq 5$, the following recurrence relation holds

$$\kappa_n = \alpha\kappa_{n-1} + (n-1)\beta^2\kappa_{n-2}.$$

4 The Probability Density Function of X_t

By Eq. (2), we have

$$
\begin{aligned}
\mathbb{E}\left(e^{iuX_t}\right) &= e^{iu\left(\mu - \frac{\sigma^2}{2}\right)t - \frac{u^2\sigma^2}{2}t - \lambda t}\left[\sum_{k=0}^{\infty} \frac{\left(\lambda t e^{iu\alpha - \frac{\beta^2 u^2}{2}}\right)^k}{k!}\right] \\
&= \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} e^{iu\left(\left(\mu - \frac{\sigma^2}{2}\right)t + k\alpha\right) - \frac{u^2}{2}\left(\sigma^2 t + k\beta^2\right)}.
\end{aligned}
$$

Taking the inverse Fourier transform, we obtain the probability density function (pdf) of X_t

$$f_{X_t}(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \frac{1}{\sqrt{2\pi(\sigma^2 t + k\beta^2)}} e^{-\frac{1}{2}\frac{\left[x - \left(\left(\mu - \frac{\sigma^2}{2}\right)t + k\alpha\right)\right]^2}{\sigma^2 t + k\beta^2}}.$$

This pdf can be derived by the law of total probability as well.

5 Calibration

There are five parameters in Merton's jump-diffusion model: $\lambda, \mu, \sigma, \alpha, \beta$. One way to determine their values to fit the market data is to solve the following system of equations:

$$\alpha\lambda t + t\left(\mu - \frac{\sigma^2}{2}\right) = m_1 \tag{3}$$

$$t\left(\sigma^2 + \lambda\left(\alpha^2 + \beta^2\right)\right) = m_2 \tag{4}$$

$$\lambda t\left(\alpha^3 + 3\alpha\beta^2\right) = m_3 \tag{5}$$

$$\lambda t\left(\alpha^4 + 6\alpha^2\beta^2 + 3\beta^4\right) = m_4 \tag{6}$$

$$\lambda t\left(\alpha^5 + 10\alpha^3\beta^2 + 15\alpha\beta^4\right) = m_5 \tag{7}$$

where the left hand side are the first five cumulants of Merton's jump-diffusion model in terms of the five parameters, which we have discussed before, and the right hand side are the first five sample cumulants obtained from market data. This system can be solved explicitly.

First we set $\tau = \sigma^2$, $\gamma = \beta^2$ to simplify the notations. If we multiply Eq. (5) by 4γ and Eq. (6) by α and then add the resulting equations together, we have

$$\alpha m_4 + 4\gamma m_3 = m_5. \tag{8}$$

Similarly, from Eqs. (3), (4), (5) and (6) we have

$$\alpha m_3 + 3\gamma(m_2 - \tau t) = m_4, \tag{9}$$

$$\alpha(m_2 - \tau t) + 2\gamma\left(m_1 - \left(\mu - \frac{\tau}{2}\right)t\right) = m_3. \tag{10}$$

Now Eqs. (8) and (9) can be thought of as a linear system of α and γ, which gives the solution

$$\alpha = \frac{3m_5\left(t\tau - m_2\right) + 4m_3m_4}{3m_4\left(t\tau - m_2\right) + 4m_3^2}, \tag{11}$$

$$\gamma = \frac{m_3m_5 - m_4^2}{3m_4\left(t\tau - m_2\right) + 4m_3^2}. \tag{12}$$

Similarly, Eqs (8) and (10) gives

$$\begin{aligned} \alpha &= \frac{m_5\left(2m_1 + t(\tau - 2\mu)\right) - 4m_3^2}{m_4\left(2m_1 + t(\tau - 2\mu)\right) + 4m_3\left(\tau t - m_2\right)}, \\ \gamma &= \frac{m_5\left(\tau t - m_2\right) + m_3m_4}{m_4\left(2m_1 + t(\tau - 2\mu)\right) + 4m_3\left(\tau t - m_2\right)}. \end{aligned} \tag{13}$$

Due to Eqs (11) and (13), we have

$$\frac{m_5\left(2m_1 + t(\tau - 2\mu)\right) - 4m_3^2}{m_4\left(2m_1 + t(\tau - 2\mu)\right) + 4m_3\left(\tau t - m_2\right)} = \frac{3m_5\left(t\tau - m_2\right) + 4m_3m_4}{3m_4\left(t\tau - m_2\right) + 4m_3^2}.$$

Solve this equation to obtain

$$\mu = \frac{m_3\left(7m_4\left(t\tau - m_2\right) - m_5\left(2m_1 + t\tau\right)\right) + m_4^2\left(2m_1 + t\tau\right) + 3m_5\left(m_2 - t\tau\right)^2 + 4m_3^3}{2\left(m_4^2 - m_3m_5\right)t}. \tag{14}$$

Now α, γ, and μ are given in Eqs. (11), (12) and (14) respectively, all in terms of τ.

From Eq. (3), we have

$$\lambda = \frac{1}{\alpha}\left[\frac{m_1}{t} - \left(\mu - \frac{\tau}{2}\right)\right]; \tag{15}$$

form Eq. (4), we have

$$\lambda = \frac{m_2/t - \tau}{\alpha^2 + \gamma}.$$

Put them together to obtain

$$\alpha\left(\frac{m_2}{t} - \tau\right) - \left(\alpha^2 + \gamma\right)\left[\frac{m_1}{t} - \left(\mu - \frac{\tau}{2}\right)\right] = 0.$$

Finally, substitute Eqs. (11), (12) and (14) into the above equation and further simplify it to obtain a quartic equation of τ

$$a_4\tau^4 + a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0 = 0,$$

where

$$\begin{aligned}
a_0 &= 16m_5m_3^6 + 48m_4^2m_3^5 - 168m_2m_4m_5m_3^4 + 8m_2\left(9m_2m_5^2 - 5m_4^3\right)m_3^3 + 225m_2^2m_4^2m_5m_3^2 - 9m_2^2m_4\left(5m_4^3 + 18m_2m_5^2\right)m_3 + 27m_2^3m_5\left(m_4^3 + m_2m_5^2\right), \\
a_1 &= \left(168m_4m_5m_3^4 + 8\left(5m_4^3 - 18m_2m_5^2\right)m_3^3 - 450m_2m_4^2m_5m_3^2 + 18m_2m_4\left(5m_4^3 + 27m_2m_5^2\right)m_3 - 27m_2^2m_5\left(3m_4^3 + 4m_2m_5^2\right)\right)t, \\
a_2 &= 9\left(8m_5^2m_3^3 + 25m_4^2m_5m_3^2 - \left(5m_4^4 + 54m_2m_5^2m_4\right)m_3 + 9m_2m_5\left(m_4^3 + 2m_2m_5^2\right)\right)t^2, \\
a_3 &= -27m_5\left(m_4^3 - 6m_3m_5m_4 + 4m_2m_5^2\right)t^3, \\
a_4 &= 27m_5^3t^4.
\end{aligned}$$

A quartic equation can be solved analytically. Once we obtain the value of τ, Eqs. (11), (12), (15) and (14) can be used to find α, γ, λ and μ, respectively. In many cases the above quartic equation has positive real roots. When there are more then one, all the positive real roots of τ will gives parameters that fit the market data very well. When no positive real root exist, an alternative choice for calibration is the method of least squares.

6 The Choice of μ Under Risk-Neutral Probability

In risk-neutral world the following identity should hold

$$\mathbb{E}(S_t) = S_0e^{rt},$$

or, equivalently,

$$\mathbb{E}(e^{X_t}) = e^{rt}.$$

On the other hand, substituting $u = -i$ into Eq. (2), we obtain

$$\mathbb{E}\left(e^{X_t}\right) = e^{\mu t + \lambda t\left(e^{\alpha + \frac{\beta^2}{2}} - 1\right)}.$$

Comparing the above two identities gives the μ under risk-neutral probability

$$\mu = r - \lambda\left(e^{\alpha + \frac{\beta^2}{2}} - 1\right).$$

7 A General Formula

Before going to option pricing, we first derivative a more general formula to help to simplify the computations. In many cases, the expectation $\mathbb{E}\left[(S_0D-K)^+\right]$ is needed when deriving the price of an option with strike price K and the initial price of underlying asset is S_0. Now we derive a general formula for this expectation with $\log D \sim N(A, B^2) \sim A+BZ$. Note that

$$\mathbb{E}\left[(S_0D-K)^+\right] = \underbrace{\mathbb{E}\left[S_0D1_{\{S_0D>K\}}\right]}_{\mathfrak{A}} - \underbrace{\mathbb{E}\left[K1_{\{S_0D>K\}}\right]}_{\mathfrak{B}},$$

where

$$\begin{aligned}\mathfrak{B} &= K\mathbb{P}(S_0D>K) \\ &= K\mathbb{P}\left(Z > \frac{\ln\frac{K}{S_0}-A}{B}\right) = KN\left(\frac{\ln\frac{S_0}{K}+A}{B}\right).\end{aligned}$$

Here we use the identity $\mathbb{P}(Z>x)=N(-x)$. On the other hand, we have

$$\begin{aligned}\mathfrak{A} &= \mathbb{E}\left[S_0e^{A+BZ}1_{\left\{Z>\frac{\ln\frac{K}{S_0}-A}{B}\right\}}\right] \\ &= S_0e^A\int_{\frac{\ln\frac{K}{S_0}-A}{B}}^{\infty} e^{Bz}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = S_0e^A\int_{\frac{\ln\frac{K}{S_0}-A}{B}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-B)^2-B^2}{2}}dz \\ &= S_0e^{A+\frac{B^2}{2}}\int_{\frac{\ln\frac{K}{S_0}-A}{B}-B}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dw = S_0e^{A+\frac{B^2}{2}}N\left(\frac{\ln\frac{S_0}{K}+A}{B}+B\right),\end{aligned}$$

where a substitution $w=z-B$ is used. To sum up, we have

$$\mathbb{E}\left[(S_0D-K)^+\right] = S_0e^{A+\frac{B^2}{2}}N\left(\frac{\ln\frac{S_0}{K}+A}{B}+B\right) - KN\left(\frac{\ln\frac{S_0}{K}+A}{B}\right). \tag{16}$$

Note that the Black-Scholes formula for a vanilla call is just e^{-rT} times the above formula with $A=(r-\sigma^2/2)T$, $B=\sigma\sqrt{T}$.

8 Pricing Formula for Vanilla Call

Perhaps the most convenient way to derive the option prices is to use the law of iterated expectations. Notice that

$$\begin{aligned}C &= e^{-rT}\mathbb{E}\left[(S_0e^{X_T}-K)^+\right] \\ &= e^{-rT}\mathbb{E}\left[\mathbb{E}\left[\left(S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T+\sum_{j=1}^{N(T)}Y_j}-K\right)^+\middle| N(T)=k\right]\right] \\ &= e^{-rT}\sum_{k=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^k}{k!}\mathbb{E}\left[\left(S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T+\sum_{j=1}^{k}Y_j}-K\right)^+\right],\end{aligned} \tag{17}$$

where

$$\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma W_T + \sum_{j=1}^{k} Y_j \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) T + k\alpha, \left(\sqrt{\sigma^2 T + k\beta^2}\right)^2\right).$$

Substituting Eq. (16) into Eq. (17) with $A = (\mu - \sigma^2/2)\, T + k\alpha, B = \sqrt{\sigma^2 T + k\beta^2}$ yields

$$C = e^{-rT} \sum_{k=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^k}{k!} \left[S_0 e^{\mu T + k\left(\alpha + \frac{\beta^2}{2}\right)} N\left(\frac{\ln \frac{S_0}{K} + \left(\mu + \frac{\sigma^2}{2}\right) T + k\left(\alpha + \beta^2\right)}{\sqrt{\sigma^2 T + k\beta^2}}\right) - KN\left(\frac{\ln \frac{S_0}{K} + \left(\mu - \frac{\sigma^2}{2}\right) T + k\alpha}{\sqrt{\sigma^2 T + k\beta^2}}\right)\right].$$