Some Notes on Fourier Transform

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1 The Definition

There are several different versions of definitions of Fourier transform. Here we adopt

$$\mathcal{F}[f(x)](t) \equiv \int_{-\infty}^{\infty} e^{itx} f(x) dx \tag{1}$$

to be the Fourier transform of $f(x)$, and

$$\mathcal{F}^{-1}[g(t)](x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} g(t) dt \tag{2}$$

to be the inverse Fourier transform of $g(t)$.

2 The Characteristic Function of the Standard Normal Distribution

Let Z be a standard normal random variable. By definition, its characteristic function is

$$\phi_Z(t) = \mathbb{E}\left(e^{itZ}\right) = \int_{-\infty}^{\infty} e^{itz} f_Z(z) dz,$$

where $f_Z(z)$ is the probability density function

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

This integral is also the Fourier transform of $f_Z(x)$. Our goal here is to show that

$$\phi_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itz} e^{-\frac{z^2}{2}} dz = e^{-\frac{t^2}{2}},$$

where we already know that

$$\phi_Z(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = 1.$$

First we apply the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, which can be obtained by the Taylor series of e^x. Now we can write

$$\phi_Z(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\cos(zt)dz + \frac{i}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\sin(zt)dz}_{0}$$

Integral by part gives

$$\begin{aligned}\phi_Z(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\cos(zt)dz + \\ &= \underbrace{\frac{\sin(zt)}{t\sqrt{2\pi}}e^{-\frac{z^2}{2}}\Bigg|_{-\infty}^{\infty}}_{0} + \frac{1}{t\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} z\sin(zt)dz.\end{aligned}$$

On the other hand, differentiating $\phi_Z(t)$ also gives

$$\phi_Z'(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\frac{\partial\cos(zt)}{\partial t}dz = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} z\sin(zt)dz.$$

Thus, we obtain the ordinary differential equation (ode)

$$\begin{aligned}\phi_Z'(t) &= -t\phi_Z(t), \\ \phi_Z(0) &= 1.\end{aligned}$$

This ode can easily be solved. Notice that

$$-t = \frac{\phi_Z'(t)}{\phi_Z(t)} = \frac{d}{dt}\ln\phi_Z(t).$$

Integrate both sides of the above equation to obtain $\phi_Z(t) = Ce^{-\frac{t^2}{2}}$ for some constant C. Substituting the initial condition, finally we have

$$\phi_X(t) = e^{-\frac{t^2}{2}}.$$

3 The Characteristic Function of the Normal Distribution

Consider a normal random variable $X \sim N(\alpha, \beta^2)$. Since X and $\alpha + \beta Z$ are identically distributed, we have

$$\begin{aligned}\phi_X(t) &= \mathbb{E}\left(e^{it(\alpha+\beta Z)}\right) \\ &= e^{it\alpha}\phi_Z(\beta t) \\ &= e^{it\alpha - \frac{\beta^2 t^2}{2}}.\end{aligned}$$

4 Fourier Inversion Theorem

In principle, taking Fourier transform and then inverse Fourier transform should makes a function unchanged, but from Eqs. (1) and (2), it seems unclear why this would happen. The Fourier inversion theorem makes sure of this. Here we will outline the proof of this theorem, that is, we are going to show that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\mathcal{F}^{-1}[f](t)dt.$$

But before we start, we should first introduce a property that will be used later:

$$\int_{-\infty}^{\infty} g(t)\left\{\mathcal{F}^{-1}[f(x)](t)\right\}dt = \int_{-\infty}^{\infty}\left\{\mathcal{F}^{-1}[g(x)](t)\right\}f(t)dt.$$

This property holds because by definition, the left hand side of the above is

$$\int_{-\infty}^{\infty} g(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}f(x)dx\right)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{itx}g(t)f(x)dtdx,$$

and the right hand side is also

$$\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}g(x)dx\right)f(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{itx}g(x)f(t)dtdx.$$

With this property in mind, now we are ready to outline the proof.

By the dominated convergence theorem, we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{ixt}\mathcal{F}^{-1}[f](t)dt &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} e^{ixt-\frac{\epsilon^2t^2}{2}}\mathcal{F}^{-1}[f](t)dt \\ &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\mathcal{F}^{-1}\left[e^{ixt-\frac{\epsilon^2t^2}{2}}\right](s)f(s)ds,\end{aligned}$$

where, from previous discussion, we already know that

$$\mathcal{F}^{-1}\left[e^{ixt-\frac{\epsilon^2t^2}{2}}\right](s) = \frac{1}{\epsilon\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-x}{\epsilon}\right)^2}.$$

Thus, we can conclude that

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} e^{ixt}\mathcal{F}^{-1}[f](t)dt \\ &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{\epsilon\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-x}{\epsilon}\right)^2}f(s)ds \to f(x).\end{aligned}$$

5 Convolution

Define the convolution of f and g:

$$F(\tau) \equiv f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(t)g(\tau - t)dt.$$

The following argument shows that $f \otimes g = \mathcal{F}^{-1}\left[\mathcal{F}[f]\mathcal{F}[g]\right]$, or equivalently, $\mathcal{F}[f \otimes g] = \mathcal{F}[f] \times \mathcal{F}[g]$:

$$\begin{aligned}
\mathcal{F}\{F(\tau)\} &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)g(\tau - t)dt \right) e^{is\tau} d\tau \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)g(\tau - t)e^{is(\tau - t)} e^{ist} dt d\tau \\
&= \int_{-\infty}^{\infty} f(t)e^{ist} \left(\int_{-\infty}^{\infty} g(\tau - t)e^{is(\tau - t)} d\tau \right) dt \\
&= \int_{-\infty}^{\infty} f(t)e^{ist} \mathcal{F}[g(\tau)]dt \\
&= \mathcal{F}[f] \times \mathcal{F}[g].
\end{aligned}$$