Carr and Madan's Option Pricing Method

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1 Introduction

The most widely used option pricing model, Black-Scholes model fails to capture some phenomenons of asset return that are usually observed in real world, such as heavy tail and higher peak. So various models are developed such as Lévy models and stochastic volatility models. It is difficult to derive a general option pricing formula for all of these models. Even some numerical procedures fail to evaluate the fair prices. A general pricing scheme is suggested by Carr and Madan (1999). In their work, a numerical method is provided to evaluate vanilla options under all kinds of model that has an analytical characteristic function of the return of underlying asset. In addition, in their algorithm the fast Fourier transform (FFT) is involved to speed up the computations. As a result, their algorithm are much more efficient than former general pricing schemes suggested by, for example, Bakshi and Madan (2000), Scott (1997).

2 Carr and Madan's Option Pricing Method

Assume that $C_T(k)$ is the price of a vanilla call option with maturity date T, strike price $K \equiv e^k$, and the price of its underlying asset at T is S_T. Let $\phi_T(v)$ stand for the characteristic function of $\log S_T$ and $q_T(v)$ its probability density function. It can be shown (Carr and Madan, 1999) that

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} \psi_T(v)\, dv, \tag{1}$$

where $\psi_T(v)$ is the Fourier transform of some modified call price $c_T(k) \equiv e^{\alpha k} C_T(k)$, which can be rewritten in terms of $\phi_T(v)$ as follows

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} c_T(k)\, dk = \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} C_T(k)\, dk \\
&= \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} e^{-rT} \int_k^\infty (e^s - e^k) q_T(s)\, ds\, dk \qquad (2) \\
&= e^{-rT} \int_{-\infty}^{\infty} q_T(s) \left[\frac{e^{k(iv+\alpha)+s}}{iv+\alpha} - \frac{e^{k(iv+\alpha+1)}}{iv+\alpha+1} \right]_{k=-\infty}^{k=s} ds \qquad (3) \\
&= \frac{e^{-rT} \phi_T(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}.
\end{aligned}$$

Here some suitable $\alpha > 0$ is chosen such that the Fourier transform of $c_T(k)$ is well defined.

If $\phi_T(v)$ (and hence $\psi_T(v)$) is known analytically, the call value given in Eq. (1) can be approximated by a quadrature

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \psi_T(v_j) \eta,$$

where $v_j \equiv \eta(j-1)$, η is sufficiently small and N is large enough to obtain accurate approximations. Substituting

$$\begin{aligned}
k_u &\equiv -b + \lambda(u-1) \quad \forall u = 1, 2, 3, \ldots, N, \\
\lambda &\equiv \frac{2\pi}{\eta N}, \\
b &\equiv \frac{N\lambda}{2},
\end{aligned}$$

into the above quadrature, we obtain

$$\begin{aligned}
C_T(k_u) &\approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-iv_j(-b+\lambda(u-1))} \psi_T(v_j) \eta \\
&= \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(u-1)} e^{ibv_j} \psi_T(v_j) \eta,
\end{aligned}$$

which can be evaluated efficiently via FFT. By inserting suitable Newton-Cotes coefficients w_j, the approximation

$$C_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} w_j e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j} \psi_T(v_j)\eta \tag{4}$$

can be even more accurate.

3 Some Issues

Two problems of this pricing scheme are mentioned in the literatures. The first one is that, there are a certain constraint on the grids. Due to the nature of FFT, $\lambda\eta$ must equals to $2\pi/N$. If η is small, the numerical integral given in Eq. (4) might converge well, but λ must be large to satisfy $\lambda\eta = 2\pi/N$. As a result, we might not get the prices of the options that are actively traded, since a sparse grid is used for k_u. To address this problem, Chourdakis (2005) involves the fractional FFT. With this modified algorithm, the constraint of $\lambda\eta = 2\pi/N$ is released. In his work, an option with strike $K \in [0.8S_0, 1.2S_0]$ is considered actively traded. Under the assumption that there are 20 grid points in this interval, a pricing algorithm with fractional FFT is more accurate compared to the original algorithm.

The other problem is raised when evaluating Eq. (1), where $e^{-ivk} = \cos vk + i \sin vk$ is involved in the integrand. In deep-out-of-the-money cases, a large k makes the integrand strongly oscillate, and hence the integral is difficult to approximate. This problem is probably mentioned and solved by Carr and Madan (2009).

As for the truncation, under the above setting the improper integral given in Eq. (1) is replaced by an definite integral on $[0, \eta N]$. Carr and Madan (1999) suggest ηN to be 1024, which is far enough in most cases as numerical results show us. Depending on the behavior of $\psi_T(v)$, ηN can be lower to enhance the efficiency of whole pricing algorithm.

References

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