How is the BOPM Related to the CONV?

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1 Convolution Method

Consider a vanilla option with strike price K and maturity date T that initiates at time 0. Let S_t stand for the price process of its underlying asset, and r, the risk-free interest rate. In the Black-Scholes framework, its fair price at time $t \in [0,T]$ is given by

$$\begin{aligned}
C_t &\equiv e^{-r(T-t)}\mathbb{E}\left[(S_T-K)^+\middle|\mathcal{F}_t\right] \\
&= S_t N\left(\frac{\ln\frac{S_t}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}N\left(\frac{\ln\frac{S_t}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W(t)} N\left(\frac{\left(\left(r-\frac{\sigma^2}{2}\right)t+\sigma W(t)\right)+\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - Ke^{-r(T-t)}N\left(\frac{\left(\left(r-\frac{\sigma^2}{2}\right)t+\sigma W(t)\right)+\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

By definition, the fair price at time $s \in [0,t)$ is

$$\begin{aligned}
C_s &\equiv e^{-r(T-s)}\mathbb{E}\left[(S_T-K)^+\middle|\mathcal{F}_s\right] \\
&= e^{-r(t-s)}\mathbb{E}\left[e^{-r(T-t)}\mathbb{E}\left[(S_T-K)^+\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right] = e^{-r(t-s)}\mathbb{E}\left[C_t\middle|\mathcal{F}_s\right]
\end{aligned}$$

or

$$\begin{aligned}
& e^{-r(t-s)}\mathbb{E}\Bigg[S_0 e^{\left(\left(r-\frac{\sigma^2}{2}\right)s+\sigma W(s)\right)+\left(\left(r-\frac{\sigma^2}{2}\right)(t-s)+\sigma(W(t)-W(s))\right)} \\
& N\left(\frac{\left(\left(r-\frac{\sigma^2}{2}\right)s+\sigma W(s)\right)+\left(\left(r-\frac{\sigma^2}{2}\right)(t-s)+\sigma(W(t)-W(s))\right)+\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
& -Ke^{-r(T-t)}N\left(\frac{\left(\left(r-\frac{\sigma^2}{2}\right)s+\sigma W(s)\right)+\left(\left(r-\frac{\sigma^2}{2}\right)(t-s)+\sigma(W(t)-W(s))\right)+\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\Bigg|\mathcal{F}_s\Bigg]
\end{aligned}$$

You don't really want to do any computation with these notations. Don't you? Setting

$$
\begin{aligned}
y &\equiv \left(\left(r-\frac{\sigma^2}{2}\right)s+\sigma W(s)\right)+\left(\left(r-\frac{\sigma^2}{2}\right)(t-s)+\sigma(W(t)-W(s))\right),\\
x &\equiv \left(\left(r-\frac{\sigma^2}{2}\right)s+\sigma W(s)\right),
\end{aligned}
$$

we can rewrite

$$
\begin{aligned}
C_s &= e^{-r(t-s)}\mathbb{E}\left[S_0e^yN\left(\frac{y+\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-Ke^{-r(T-t)}N\left(\frac{y+\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\middle|\mathcal{F}_s\right]\\
&= e^{-r(t-s)}\int_{-\infty}^{\infty}\left(S_0e^yN\left(\frac{y+\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-Ke^{-r(T-t)}N\left(\frac{y+\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right)\\
&\qquad\cdot\frac{1}{\sigma\sqrt{2\pi(t-s)}}e^{-\frac{1}{2\sigma^2}\frac{\left[y-x-\left(r-\frac{\sigma^2}{2}\right)(t-s)\right]^2}{t-s}}dy.
\end{aligned}
$$

This integral is the start point of the convolution pricing method. Although we already know that

$$
\begin{aligned}
C_s &= S_0e^xN\left(\frac{x+\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-s)}{\sigma\sqrt{T-s}}\right)-Ke^{-r(T-s)}N\left(\frac{x+\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-s)}{\sigma\sqrt{T-s}}\right)\\
&= e^{-r(t-s)}\int_{-\infty}^{\infty}\left(S_0e^yN\left(\frac{y+\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-Ke^{-r(T-t)}N\left(\frac{y+\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right)\\
&\qquad\cdot\frac{1}{\sigma\sqrt{2\pi(t-s)}}e^{-\frac{1}{2\sigma^2}\frac{\left[y-x-\left(r-\frac{\sigma^2}{2}\right)(t-s)\right]^2}{t-s}}dy,
\end{aligned}
$$

it is difficult to evaluate this integral by common techniques of integration.

2 The Binomial Option Pricing Model

We start with the last integral. The binomial option pricing model uses the following approximation

$$
C_s(x)\approx e^{-r(t-s)}\left[pC_t(x^+)+(1-p)C_t(x^-)\right],
$$

where x^+ is a little bit more than x, and x^- is a little bit less than x^-. There are many different choices of x^+, x^- and p. For example, one choice is that

$$
\begin{aligned}
x^+ &= x+\sigma\sqrt{t-s},\\
x^- &= x-\sigma\sqrt{t-s},\\
p &= \frac{e^{r(t-s)}-e^{-\sigma\sqrt{t-s}}}{e^{\sigma\sqrt{t-s}}-e^{-\sigma\sqrt{t-s}}}.
\end{aligned}
$$

It can be shown by central limit theorem that under this setting the approximation converges, although only with order 1. To go further to analyze the error, one should consider the difference

$$C_s(x) - e^{-r(t-s)} \left[pC_t(x^+) + (1-p)C_t(x^-) \right],$$

or more precisely,

$$\begin{aligned}
&S_0 e^x N\left(\frac{x + \ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-s)}{\sigma\sqrt{T-s}}\right) - Ke^{-r(T-s)} N\left(\frac{x + \ln\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-s)}{\sigma\sqrt{T-s}}\right) \\
&-e^{-r(t-s)}\left[p\left(S_0 e^{x+\sigma\sqrt{t-s}} N\left(\frac{x + \sigma\sqrt{t-s} + \ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)} N\left(\frac{x + \sigma\sqrt{t-s} + \ln\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right)\right. \\
&\quad\left.(1-p)\left(S_0 e^{x-\sigma\sqrt{t-s}} N\left(\frac{x - \sigma\sqrt{t-s} + \ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)} N\left(\frac{x - \sigma\sqrt{t-s} + \ln\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right)\right].
\end{aligned}$$