Box-Muller Transform

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1 Inverse Transform Sampling

Let U be a random variable that uniformly distributed on $(0,1]$. The following argument shows that $F_Z^{-1}(U)$ and Z have the same distribution.

$$\begin{aligned}
y &= \mathbb{P}(y < U) \qquad \forall y \in (0,1] \\
F_Z(z) &= \mathbb{P}(F_Z(z) < U) \\
&= \mathbb{P}(z < F_Z^{-1}(U)) \\
&= \mathbb{P}(z < Z) \\
\therefore F_Z^{-1}(U) &\sim Z
\end{aligned}$$

2 Box-Muller Transform

Box-Muller transform is used to generate i.i.d. standard normal random numbers X and Y. Assume U_1, U_2 are i.i.d. random variables uniformly distributed on $(0,1]$, and $R = \sqrt{X^2+Y^2}$, $\quad \Theta = \arctan(Y/X)$. We have

$$\begin{aligned}
f_{R,\Theta}(r,\theta) &= f_{X,Y}(x,y)\frac{\partial(x,y)}{\partial(r,\theta)} \\
&= \frac{1}{2\pi} r e^{-\frac{r^2}{2}} \mathbf{1}_{\{r>0,\theta\in(0,2\pi]\}} \\
\therefore f_\Theta(\theta) &= \int_0^\infty \frac{1}{2\pi} r e^{-\frac{r^2}{2}} dr = \frac{1}{2\pi}\left[-e^{-\frac{r^2}{2}}\right]_0^\infty \\
&= \frac{1}{2\pi}\mathbf{1}_{\{\theta\in(0,2\pi]\}} \\
\therefore f_R(r) &= r e^{-\frac{r^2}{2}} \mathbf{1}_{\{r>0\}}
\end{aligned}$$

$$\begin{aligned}
F_R(z) &= \int_0^z r e^{-\frac{r^2}{2}} dr = 1 - e^{-\frac{z^2}{2}} \, (= w) \\
F_R^{-1}(w) &= \sqrt{-2\ln(1-w)}
\end{aligned}$$

$$\therefore \begin{cases} R \sim \sqrt{-2\ln U_1} \\ \Theta \sim 2\pi U_2 \end{cases}, \quad \begin{cases} X \sim \sqrt{-2\ln U_1}\cos(2\pi U_2) \\ Y \sim \sqrt{-2\ln U_1}\sin(2\pi U_2) \end{cases}.$$

3 Marsaglia Polar Method

Let $\mathcal{U}_1$,$\mathcal{U}_2$ be random variables with joint density function

$$f_{\mathcal{U}_1,\mathcal{U}_2}(u,v) = \frac{1}{\pi}\mathbf{1}_{\{u^2+v^2\leq 1\}}.$$

The following argument shows that $\mathcal{U}_1^2 + \mathcal{U}_2^2$ uniformly distributed on $[0,1]$. Let $\mathbf{R}$ and $\mathbf{\Theta}$ stand for $\sqrt{\mathcal{U}_1^2 + \mathcal{U}_2^2}$ and $\arctan(\mathcal{U}_2/\mathcal{U}_1)$, respectively. Note that

$$\begin{aligned}
f_{\mathbf{R},\mathbf{\Theta}}(r,\theta) &= f_{\mathcal{U}_1,\mathcal{U}_2}(u,v)\frac{\partial(u,v)}{\partial(r,\theta)} \\
&= \frac{r}{\pi}\mathbf{1}_{\{0\leq r\leq 1, 0\leq\theta<2\pi\}} \\
\therefore f_{\mathbf{R}}(r) &= \int_0^{2\pi} f_{\mathbf{R},\mathbf{\Theta}}(r,\theta)d\theta = 2r\cdot\mathbf{1}_{\{0\leq r\leq 1\}} \\
\therefore f_{\mathbf{R}^2}(a) &= f_{\mathbf{R}}(r)\frac{1}{2r} = \mathbf{1}_{\{0\leq r\leq 1\}}.
\end{aligned}$$

Thus, similar to Box-Muller transform, first we generate random variables $\mathcal{U}_1$,$\mathcal{U}_2$ uniformly distributed on a unit disk. Then $s \equiv \mathbf{R}^2 = \mathcal{U}_1^2 + \mathcal{U}_2^2$ is evaluated. Since $s \sim U_1$ holds, we have $\sqrt{-2\ln s} \sim \sqrt{-2\ln U_1}$. In addition, obviously we have

$$\begin{aligned}
(\cos(2\pi U_2), \sin(2\pi U_2)) &\sim (\cos\mathbf{\Theta}, \sin\mathbf{\Theta}) \\
&= \left(\frac{\mathcal{U}_1}{\mathbf{R}}, \frac{\mathcal{U}_2}{\mathbf{R}}\right) = \left(\frac{\mathcal{U}_1}{\sqrt{s}}, \frac{\mathcal{U}_2}{\sqrt{s}}\right).
\end{aligned}$$

Since $\mathbf{R}$ and $\mathbf{\Theta}$ are independent, we know that

$$\left(\mathcal{U}_1\sqrt{\frac{-2\ln s}{s}}, \mathcal{U}_2\sqrt{\frac{-2\ln s}{s}}\right)$$

is a standard bivariate normal random vector