A NONZERO-SUM GAME APPROACH TO CONVERTIBLE BONDS：TAX BENEFIT,BANKRUPTCY COST, AND EARLY/LATE CALLS

Recall

- 原先在亨利學長論文上的delay call：

1. 如果沒有稅盾和破產成本的話，V=SB+CB+E

   此時max(Equity value)=min(CB Value)

2. 加入稅盾和破產成本後，SB+CB+E=VL=VA+TB-TC

   此時max(Equity value)=\=min(CB Value)

   EX：假設到期日時，CP符合

   $$\underline{F_C(1+(1-\tau)C_C\Delta t) < CP < F_C(1+C_C\Delta t)}$$

   贖回可轉換公司債會降低公司稅盾，贖回動作可能導致股東權益較贖回前更低，造成公司決定不贖回可轉換公司債

   →贖回延遲

Equity

- 到期日
  - 若公司贖回可轉換公司債

$$Equity_{n,call}(y\%) = \max(V_n - N_B F_B(1+(1-\tau)C_B \Delta t) - (1-y\%)N_C CP, 0)$$

max(公司價值－普通債價值－剩餘未轉換CB贖回價值,0) (3

  - 若公司不贖回可轉換公司債

$$Equity_{n,uncall}(y\%) = \max(V_n - N_B F_B(1+(1-\tau)C_B \Delta t) - (1-y\%)N_C F_C(1+(1-\tau)C_c \Delta t), 0)$$

max(公司價值－普通債價值－剩餘未轉換CB價值,0) (3

  - 若可轉換公司債先前已被贖回

$$Equity_{n,called}(y\%) = \max(V_n - N_B F_B(1+(1-\tau)C_B \Delta t), 0)$$

max(公司價值－普通債價值,0) (3

- 股票價值計算

$$Stock_{n,*} = \begin{cases} \dfrac{Equity_{n,*}(y\%)}{N_0 + y\% \times N_C \times \gamma} & if \quad Equity_{n,*}(y\%) > 0 \\ 0 \quad , & Otherwise \end{cases}$$

$$* = call, uncall, called$$

(3.

EARLY/LATE CALLS

- When the conversion value is signficantly small than the call price ,which is known as an out of money call or early call.
- If the conversion value is signficantly large than the call price ,which is known as an in of money call or late call.

- 本篇的game: a convertible bondholder v.s. a sharehoder
- 只存在一張可轉債
- 假設 company value = V

  call price = K

  conversion factor = λ
- When $\lambda V>K$ => late call and $\lambda V<K$ => early call

The model

- Let us begin with a probability space (Ω,F,P) on which is given an exogenous standard Brown motion W_t.
- Assume that the asset of the company are tradeable ,so we can construct the risk-netural probability Q. Under it , the discounted total value of the company is governed by a martingale process.

The company produces a cash flow continuously as follows :

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dW_t$$, where μ, σ are constants.

All the investors in the market are supposed to have a power - function utility given by

$$E^P[\int_0^\infty e^{-\beta t} \frac{\varepsilon_t^{1-\gamma}}{1-\gamma} dt] \qquad (2.2)$$

the investor choose ε_t at time t , $\gamma \in (0,1)$ is the coefficient of relative risk aversion.

$$dB_t = r_t B_t dt \ , \ B_0 = 1$$

$$Q(A) = E^P[\exp(-\sigma\gamma W_t - \frac{1}{2}\sigma^2\gamma^2 t) \cdot 1_A]$$

$$V_t = E^Q[\int_t^\infty e^{-r(u-t)} \delta_u du \mid \delta_t] = \frac{\delta_t}{r-(\mu-\sigma^2\gamma)}$$

Denote $\delta := \delta_t / V_t$ .Then $$\frac{dV_t}{V_t} = (r-\delta)dt + \sigma dW_t^Q \qquad (2.3)$$

where $W_t^Q = W_t + \gamma\sigma t$ is a standard Borwnian motion under Q.

假設

Face value	**P**
Coupon rate	c
Conversion factor	λ $(0 \leq \lambda \leq 1)$
Call price	K
Corporate tax rate	κ(kappa)
Delta t	dt
破產成本	ρ
1.$1-\rho \geq$ λ(避免bondholder 總是在公司default 前 convert)	
2.公司初始資產完全只有equity(沒有普通債, unlevered asset)	
3.公司只發行一張可轉債，且這張可轉債是永續的(perpetual)	
4. Γ={$\tau \geq 0; \tau\ is\ a\ stopping\ time\ adaptive\ with\ respect\ to\ \{F_t\}$}	
5.Denote the conversion,bankruptcy and call time by τ_{con}、τ_b、$\tau_{call} \in \Gamma$	

- if $\tau_{con} < \tau_b \wedge \tau_{cal} = \min(\tau_b, \tau_{cal})$,then the conversion from the bondholder will leave $(1-\lambda)V_{con}$ to the shareholder.However, if $\tau_{con} > \tau_b \wedge \tau_{cal}$,there exist three possibilities:

$$V_{\tau_b \wedge \tau_{cal}} \leq K \,, K < V_{\tau_b \wedge \tau_{cal}} \leq \frac{K}{\lambda} \;\;, V_{\tau_b \wedge \tau_{cal}} > \frac{K}{\lambda}$$

Let

$$h(V) = min\ \left((V-K)^+, (1-\lambda)V\right) = \begin{cases} 0, \;\; V < K \\ V-K, \;\; K \leq V < \frac{K}{\lambda} \\ (1-\lambda)V, V \geq \frac{K}{\lambda} \end{cases}$$

and $g(V) = \begin{cases} (1-\rho)V, \;\; V < K \\ V - h(V), \;\; V \geq K \end{cases}$

- Before $\tau_{con} \wedge \tau_b \wedge \tau_{cal}$,the bondholder keeps receiving coupons at a rate of Pc until one of the capital-structure-change events occurs.
- The shareholder has an obligation to serve the coupon payments to the debt holder continuously.
- The company generates a cash flow amount of $\delta V_t dt$ by its operation at time t. After coupon obligation, the remaining cash flow is $(\delta V_t - (1 - \kappa)cP)dt$ .The quantity $\delta V_t - (1 - \kappa)cP$ may be negative.

- Denote $D(V;\tau_b,\tau_{cal};\tau_{con})$ and $\mathrm{E}(V;\tau_b,\tau_{cal};\tau_{con})$ to be the respective bond and equity value when the company is worth V and the policies $\tau_b, \tau_{cal}\ and\ \tau_{con}$ are fixed.There,we have

$$
\begin{aligned}
&E(V;\tau_b,\tau_{cal};\tau_{con}) \\
&= E^Q[\int_0^{\tau_{con}\wedge\tau_b\wedge\tau_{cal}} e^{-rt}(\delta V_t-(1-\kappa)cP)dt + e^{-r(\tau_b\wedge\tau_{cal})}h(V_{\tau_b\wedge\tau_{cal}})1_{\{\tau_{con}\ge\tau_b\wedge\tau_{cal}\}} \\
&\qquad + e^{-r\tau_{con}}(1-\lambda)V_{\tau_{con}}1_{\{\tau_{con}<\tau_b\wedge\tau_{cal}\}} \mid V_0=V]\cdots(2.4)
\end{aligned}
$$

$$
\begin{aligned}
&D(V;\tau_b,\tau_{cal};\tau_{con}) \\
&= E^Q[\int_0^{\tau_{con}\wedge\tau_b\wedge\tau_{cal}} e^{-rt}cPdt + e^{-r\tau_{con}}\lambda V_{\tau_{con}}1_{\{\tau_{con}<\tau_b\wedge\tau_{cal}\}} \\
&\qquad + e^{-r(\tau_b\wedge\tau_{cal})}g(V_{\tau_b\wedge\tau_{cal}})1_{\{\tau_{con}\ge\tau_b\wedge\tau_{cal}\}} \mid V_0=V]\cdots(2.5)
\end{aligned}
$$

- The bondholder and the shareholder wish to maximize the values of their respective holdings.This creates a two-person game such that

$$\tau_{con}^{*} = \arg\max_{\tau_{con} \in \Gamma} D(V; \tau_{b}^{*}, \tau_{cal}^{*}; \tau_{con}) \cdots (2.6)$$

$$(\tau_{b}^{*}, \tau_{cal}^{*}) = \arg\max_{\tau_{b}, \tau_{cal} \in \Gamma} D(V; \tau_{b}, \tau_{cal}; \tau_{con}^{*}) \cdots (2.7)$$

- Given the unlevered company value V at time 0.

$$E + D = V + E^{Q}[\int_{0}^{\tau_b \wedge \tau_{cal} \wedge \tau_{con}} e^{-rt} \kappa c P dt] - E^{Q}[e^{-r\tau_b} \rho V_{\tau_b} 1_{\{\tau_b < \tau_{cal} \wedge \tau_{con}\}}]$$

Variational Inequalities Formulation

- To solving the game (2.6 & 2.7)for Nash Equilibria
- Let

$$U = \{f : [0,\infty) \to [0,\infty) \mid \mathrm{f}(0) = 0, \mathrm{f} \text{ continuous. There are two disjoint finite sets } \mathrm{N}_\mathrm{f}^1, \mathrm{N}_\mathrm{f}^2 \text{ such that } \mathrm{f} \in \mathrm{C}^1([0,\infty) \setminus \mathrm{N}_\mathrm{f}^1) \cap \mathrm{C}^2([0,\infty) \setminus \mathrm{N}_\mathrm{f}^1 \cup \mathrm{N}_\mathrm{f}^2). \text{The first derivative } \mathrm{f}' \text{ is bounded on } [0,\infty) \setminus \mathrm{N}_\mathrm{f}^1. \quad \mathrm{f}'(\mathrm{a}\pm) = \lim_{\mathrm{v}\to\mathrm{a}\pm} \mathrm{f}'(\mathrm{v}) \text{ and } \mathrm{f}''(\mathrm{b}\pm) = \lim_{\mathrm{v}\to\mathrm{b}\pm} \mathrm{f}''(\mathrm{v}) \text{ exists and are finite for } \mathrm{a} \in N_f^1 \text{ and } \mathrm{b} \in N_f^2\}.$$

- Define an operator L such that it maps any function $f \in U$ into a new function such that

$$Lf(v) = -\frac{1}{2}\sigma^2 v^2 \frac{d^2 f(v)}{dv^2} - (r-\delta)v\frac{df(v)}{dv} + rf(v)$$

- for any $v \in [0,\infty) \setminus (N_f^1 \cup N_f^2)$ .It is a delicate issue to define Lf at $v \in N_f^1 \cup N_f^2$ because the first or second derivatives of f do not exist on those points.In the paper we adopt the following convertion such that

$$\mathrm{Lf(v)} := \begin{cases} -\infty \ ,\mathrm{if}\ \mathrm{v} \in \mathrm{N_f^1} \ \ \text{and}\ \ \mathrm{f}^{'}(v+) > f^{'}(v-) \\ \ \ \infty \ ,\mathrm{if}\ \mathrm{v} \in \mathrm{N_f^1} \ \ \text{and}\ \ \mathrm{f}^{'}(v+) < f^{'}(v-) \end{cases}$$

and $Lf(v) := Lf(v-)$ for $v \in N_f^2 \setminus N_f^1$

Conditions 1~6

- Suppose that d and e represent the optimal bond and equity value functions, respectively.There two observations require that the function d and e should satisfy:

1. $d(V) \geq \lambda V\ and\ e(V) \geq h(V)\ for\ all\ V \geq 0$

- Define
  $S_E := cl(\{V \in [0,\infty): e(V) = h(V), Le(V) > (\delta V - (1-\kappa)Pc)\})$
  $S_D := cl(\{V \in [0,\infty): d(V) = \lambda V, Ld(V) > Pc\})$

2. $If\ V \in S_D, then\ e(V) = (1-\lambda)V$
3. $If\ V \in S_E, then\ d(V) = g(V)$

Conditions 1~6

4. $N_d^1 \subset \mathrm{S_D}$.On the set $\mathrm{S_E^C} := [0, \infty) \setminus S_E$,

   the function d satisfies $\min\{\mathrm{d(V)} - \lambda \mathrm{V}, \mathrm{Ld(V)} - \mathrm{cP}\} = 0$

5. $N_e^1 \subset S_D \cup \{K/\lambda\}.e^{'}(K \setminus \lambda+)$and the equality holds if and only if $\mathrm{K}/\lambda \in \mathrm{S_E^C}$.

   On the set $\mathrm{S_D^C} := [0, \infty) \setminus S_D$,

   the function e satisfies $\min\{\mathrm{e(V)} - \mathrm{h(V)}, \mathrm{Le(V)} - (\delta \mathrm{V} - (1-\kappa)\mathrm{cP})\} = 0$

Finally,we assume that

6. $S_D \cap S_E = \emptyset$

Theorem 3.1

Suppose that there exists a pair of functions $\{d^*, e^*\} \in U$ that satisfy Condiction 1-6. Define

$$\tau^*_{con} = \inf\{t \geq 0 : V_t \in S_D\}$$

$$\tau^*_b = \inf\{t \geq 0 : V_t \in S_E \cap [0, K]\}$$

$$\tau^*_{cal} = \inf\{t \geq 0 : V_t \in S_E \cap (K, \infty)\}$$

Then,these stopping time constitute a Nash Equilibrium for game (2.6 and 2.7).

Furthermore,

$$d^*(V) = D(V; \tau^*_b, \tau^*_{cal}; \tau^*_{con}) \text{ and } e^*(V) = E(V; \tau^*_b, \tau^*_{cal}; \tau^*_{con})$$

where D and E are defined as in (2.4) and (2.5)

Theorem 3.2

Suppose that two functions e*and d* in U solve the system of variational inequalities 1-6.The following conclusion holds:

(i)There exists a unique $V_{con}^{*} \in [(cP/(\delta\lambda),\infty)$ such that $\mathrm{S_D} = [\mathrm{V_{con}^{*}},\infty)$

(ii)There exists a unique $V_b^{*} \in (0,\min\{K,(1-\kappa)cP/\delta\}]$ such that

$S_E \cap [0,K] = [0,V_b^{*}]$

(iii)A necessary condiction of $S_E \cap (K,\infty) \neq \varnothing$ is that $K \leq (1-\kappa)cP/\delta$

(iv)Moreover, if $S_E \cap (K,\infty) \neq \varnothing$, then $\mathrm{K}/\lambda \in S_E \cap (K,\infty)$ and there exist unique $V_{cal,1}^{*}(K,K/\lambda]$ and $V_{cal,2}^{*}[K/\lambda,(1-\kappa)cP/\lambda\delta]$ such that

$S_E \cap (K,\infty) = [V_{cal,1}^{*},V_{cal,2}^{*}]$

Nash Equilbrium

- For any three real numbers such that $0 < b \leq v \leq d$,let ς be the first passage time of Vt across boundaries V=b or V=d,i.e., $\varsigma = \inf\{t \geq 0: V_t \leq b \text{ } or \text{ } V_t \geq d\}$
- Define function p and q be the present values of two Arrow-Debreu securities , paying one dollar on the event Vς=b and Vς=d ,respectively.In other words,

$$p(V;b,d) = E[e^{-r\varsigma} 1_{\{V_\varsigma = b\}} \mid V_0 = v] \text{ and } q(V;b,d) = E[e^{-r\varsigma} 1_{\{V_\varsigma = d\}} \mid V_0 = v]$$

Nash Equilbrium

- Under the specification of geometric Brownian montion(2.3),both of then admit closed-form expressions

$$p(V;b,d)=\frac{d^{\beta+\gamma}-v^{\beta+\gamma}}{d^{\beta+\gamma}-b^{\beta+\gamma}}\left(\frac{b}{v}\right)^{\gamma}\ \text{and}\ q(V;b,d)=\frac{v^{\beta+\gamma}-b^{\beta+\gamma}}{d^{\beta+\gamma}-b^{\beta+\gamma}}\left(\frac{d}{v}\right)^{\gamma}$$

where

$$\beta=\frac{-(r-\delta-\sigma^2/2)+\Delta}{\sigma^2},\ \gamma=\frac{(r-\delta-\sigma^2/2)+\Delta}{\sigma^2}\quad (4.1)\text{and}\ \Delta=\sqrt{(r-\delta-\sigma^2/2)+2r\sigma^2}$$

- According Theorem 3.2,there are only two possibilities :the shareholder never calls the debt back when K is sufficiebtly and he may issue a call for small K.

L operator

The company follows $dv = (r-\delta)vdt + \sigma v dW_t^Q$

Suppose there exists a claim this company and this claim continuously pays coupon (or dividend) C per year

By Ito's lemma

$$df(v) = \frac{\partial f}{\partial v}dv + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}(\mathrm{dv})^2$$

$$= [\frac{\partial f}{\partial v}(r-\delta)v + \frac{1}{2}\sigma^2 v^2 \frac{\partial^2 f}{\partial v^2}]dt + \sigma v dW_t^Q$$

L operator

建造一無風險投資組合S為1單位 f 和 $-\frac{\partial f}{\partial v}$ 單位標的物

$$S = -\frac{\partial f}{\partial v} v + f(v)$$

$$dS = rSdt = -\frac{\partial f}{\partial v} dv + df(v) + Cdt - \delta v \frac{\partial f}{\partial v} dt$$

$$= -\frac{\partial f}{\partial v}((r-\delta)vdt + \sigma v dW_t^Q) + [\frac{\partial f}{\partial v}(r-\delta)v + \frac{1}{2}\sigma^2 v^2 \frac{\partial^2 f}{\partial v^2} + C - \delta v \frac{\partial f}{\partial v}]dt + \frac{\partial f}{\partial v}\sigma V dW_t^Q$$

$$= [\frac{1}{2}\sigma^2 v^2 \frac{\partial^2 f}{\partial v^2} + C - \delta v \frac{\partial f}{\partial v}]dt$$

$$(f - v\frac{\partial f}{\partial v})rdt = [\frac{1}{2}\sigma^2 v^2 \frac{\partial^2 f}{\partial v^2} + C - \delta v \frac{\partial f}{\partial v}]dt$$

$$\Rightarrow rf = (r-\delta)v\frac{\partial f}{\partial v} + \frac{1}{2}\sigma^2 v^2 \frac{\partial^2 f}{\partial v^2} + C$$

$$\Rightarrow -(r-\delta)v\frac{\partial f}{\partial v} - \frac{1}{2}\sigma^2 v^2 \frac{\partial^2 f}{\partial v^2} + rf = C$$

Appendix A

已知LD(v) = $-\frac{1}{2}\sigma^2 v^2 \frac{d^2}{dv^2} D(v) - (r-\delta) v \frac{d}{dv} D(v) + rD(v) = Pc$

By The Euler - Cauchy ODE

Let $\mathrm{v} = \mathrm{e}^{\mathrm{t}}$, $\mathrm{t} = \ln(\mathrm{x})$, $\mathrm{Z} = \frac{\mathrm{d}}{\mathrm{dt}}$

then $\mathrm{D}' = \frac{\mathrm{dD}}{\mathrm{dv}} = \frac{dD}{dt}\frac{dt}{dv} = \frac{1}{v}\frac{dD}{dt} = \frac{1}{v} ZD$

$$\mathrm{D}'' = \frac{d^2 D}{dv^2} = \frac{d}{dv}\left(\frac{1}{v}\frac{dD}{dt}\right) = -\frac{1}{v^2}\frac{dD}{dt} + \frac{1}{v}\frac{d\frac{dD}{dt}}{dv} = -\frac{1}{v^2}\frac{dD}{dt} + \frac{1}{v}\frac{d\frac{dD}{dt}}{dt}\frac{dt}{dv}$$

$$= -\frac{1}{v^2}\frac{dD}{dt} + \frac{1}{v^2}\frac{d^2 D}{dt^2} = \frac{1}{v^2} Z(Z-1)D$$

$$\{Z(Z-1) - \frac{(r-\delta)}{(-\frac{1}{2}\sigma^2)} Z + \frac{r}{(-\frac{1}{2}\sigma^2)}\} D(v) = \frac{Pc}{(-\frac{1}{2}\sigma^2)}$$

Appendix A

1.齊次解

$$\Rightarrow m^2 - \frac{(\mathrm{r} - \delta - \frac{\sigma^2}{2})}{(-\frac{1}{2}\sigma^2)} m + \frac{r}{(-\frac{1}{2}\sigma^2)} = 0$$

$$\Rightarrow \mathrm{m} = \frac{(\mathrm{r} - \delta - \frac{\sigma^2}{2}) \pm \sqrt{(\mathrm{r} - \delta - \frac{\sigma^2}{2})^2 + 2r\sigma^2}}{\sigma^2} = \beta \quad and \quad -\gamma$$

$$\Rightarrow \mathrm{D(v)}_{\mathrm{h}} = c_1 e^{\beta t} + c_2 e^{-\gamma t} = c_1 v^{\beta} + c_2 v^{-\gamma}$$

2.特解

令$D(v)_h = A$ 帶回 LD(v) 可以得 $A = \frac{Pc}{r}$

3.$D(v)^* = D(v)_h + D(v)_p = \frac{Pc}{r} + c_1 v^{\beta} + c_2 v^{-\gamma}$

Appendix A

1.E(v)的其次解與D(v)的其次解解法一模一樣

2.假設$\mathrm{E(v)_p} = \mathrm{A}v + B$帶回LE(v)

$$-(\mathrm{r}-\delta)\mathrm{vA} + r(\mathrm{Av}+\mathrm{B}) = \delta A v + rB = \delta v - (1-\kappa)Pc$$

$$\text{Hence}, \mathrm{A} = 1, B = -\frac{(1-\kappa)Pc}{r}$$

$$\Rightarrow \mathrm{E(v)_p} = v - \frac{(1-\kappa)Pc}{r}$$

3.$E(v)^* = E(v)_p + E(v)_h = v - \dfrac{(1-\kappa)Pc}{r} + c_3 v^{\beta} + c_4 v^{-\gamma}$

No Voluntary Calls

Theorem 4.1

There are exist a critical value K_1.When $K \geq K_1$, we can find unique V_b^* and V_{con}^* such that in a Nash equilibrium, the bondholder should convert at

$$\tau_{con}^* = \inf\{t \geq 0 : V_t \geq V_{con}^*\}$$

and the shareholder never calls snd should announce backruptcy at the monent

$$\tau_b^* = \inf\{t \geq 0 : V_t \geq V_b^*\}$$

Furthermore, the optimal equity and bond values are given by

$$(\mathrm{E}^*(\mathrm{V}), \mathrm{D}^*(\mathrm{V})) = \begin{cases} (0, (1-\rho)\mathrm{V}) & ,\text{if } \mathrm{V} \leq \mathrm{V}_\mathrm{b}^* \\ (\mathrm{E}_1(\mathrm{V}; \mathrm{V}_\mathrm{b}^*, \mathrm{V}_\mathrm{con}^*), \mathrm{D}_1(\mathrm{V}; \mathrm{V}_\mathrm{b}^*, \mathrm{V}_\mathrm{con}^*)) & ,\text{if } \mathrm{V}_\mathrm{b}^* < V \leq \mathrm{V}_\mathrm{con}^* \\ ((1-\lambda)\mathrm{V}, \lambda\mathrm{V}) & ,\text{if } \mathrm{V} \geq \mathrm{V}_\mathrm{con}^* \end{cases}$$

$$\text{where } \mathrm{E}_1(\mathrm{V}; \mathrm{V}_\mathrm{b}^*, \mathrm{V}_\mathrm{con}^*) = V - \frac{(1-\kappa)Pc}{r} + \left(\frac{(1-\kappa)Pc}{r} - \mathrm{V}_\mathrm{b}^*\right) p(V; V_b^*, V_{con}^*)$$
$$+ \left(\frac{(1-\kappa)Pc}{r} - \lambda V_{con}^*\right) q(V; V_b^*, V_{con}^*)$$

$$\mathrm{D}_1(\mathrm{V}; \mathrm{V}_\mathrm{b}^*, \mathrm{V}_\mathrm{con}^*) = \frac{Pc}{r} + \left((1-\rho)\mathrm{V}_\mathrm{b}^* - \frac{Pc}{r}\right) p(V; V_b^*, V_{con}^*)$$
$$+ \left(\lambda V_{con}^* - \frac{Pc}{r}\right) q(V; V_b^*, V_{con}^*)$$

No Voluntary Calls

For any $V \in (V_b^*, V_{con}^*) = S_E^C \cap S_D^C$, the game will not be stopped and then the equity and bond value functions should satisfy the ODEs

$$LE^*(V) = \delta V - (1-\kappa)Pc \text{ and } LD^*(V) = Pc$$

$$\Rightarrow D^*(V) = \frac{Pc}{r} + c_1 V^{\beta} + c_2 V^{-\gamma} \text{ and } E^*(V) = V - \frac{(1-\kappa)Pc}{r} + c_3 V^{\beta} + c_4 V^{-\gamma}$$

We can fix the four constants $c_i, 1 \le i \le 4$, making use of the boundary conditions

$$\begin{cases} D^*(V_{con}^*) = \lambda V_{con}^* \\ E^*(V_{con}^*) = (1-\lambda) V_{con}^* \end{cases} \text{ and } \begin{cases} D^*(V_b^*) = (1-\rho) V_b^* \\ E^*(V_b^*) = 0 \end{cases}$$

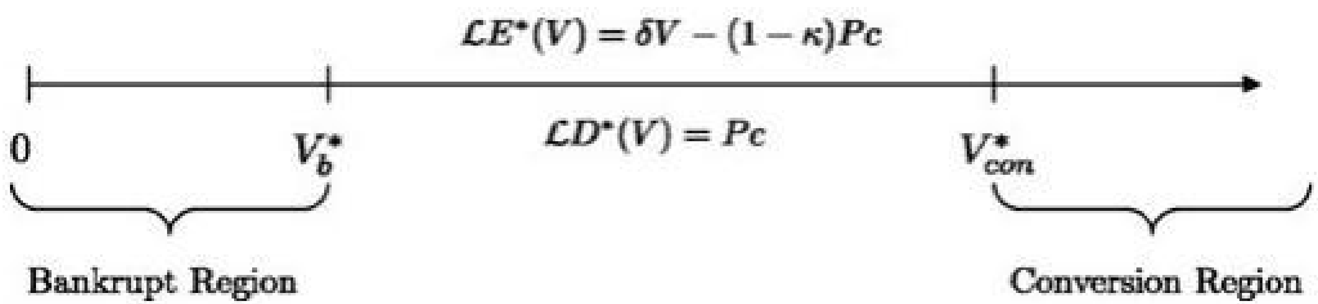

FIGURE 4.1. The optimal conversion and bankruptcy regions when $K > K_1$. The shareholder declares bankruptcy on $S_E = [0, V_b^*]$ and the bondholder converts on $S_D = [V_{con}^*, +\infty)$.

No Voluntary Calls

According to Condiction 4 and 5 , E^* should be differentiable on $S_D^C = [0, V_{con}^*)$ and D^* should be differentiable on $S_E^C = (V_b^*, \infty)$.

This amounts to requiring

$$\left.\frac{\mathrm{dE}^*(V)}{dV}\right|_{V=V_b^*} = \left.\frac{dE_1^*(V;V_b^*,V_{con}^*)}{dV}\right|_{V=V_b^*+} = 0 \cdots (4.2)$$

where 0 is the left derivative of E^* at V_b^* because $\mathrm{E}^*(V) = 0$ for all $V \in [0, V_b^*)$

$$\left.\frac{\mathrm{dD}^*(V)}{dV}\right|_{V=V_{con}^*} = \left.\frac{dD_1^*(V;V_b^*,V_{con}^*)}{dV}\right|_{V=V_{con}^*-} = \lambda \cdots (4.3)$$

where λ is the right derivative of D^* at V_{con}^* because $\mathrm{D}^*(V) = \lambda V$ for all $V \in [V_{con}^*, \infty)$ .Then we can solve uniquely V_b^* and V_{con}^*.

Lemma D.1

$$(\mathrm{D.1})\ \mathrm{V}_{\mathrm{b}}^{*} = \frac{(1-\kappa)Pc}{r} \cdot \frac{\gamma + \beta(V_b^{*}/V_{con}^{*})^{\beta+\gamma} - (\beta+\gamma)(V_b^{*}/V_{con}^{*})^{\beta}}{(\gamma+1) + (\beta-1)(V_b^{*}/V_{con}^{*})^{\beta+\gamma} - \lambda(\beta+\gamma)(V_b^{*}/V_{con}^{*})^{\beta-1}}$$

$$\begin{aligned}(D.2)0 = {} & x(\beta + \gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma})((\gamma+1) + (\beta-1)x^{\beta+\gamma} - \lambda(\beta+\gamma)x^{\beta-1}) \\ & - (1-\kappa)(\gamma + \beta x^{\beta+\gamma} - (\beta+\gamma)x^{\beta})(\lambda(\beta-1) + \lambda(\gamma+1)x^{\beta+\gamma} \\ & - (1-\rho)(\beta+\gamma)x^{\gamma+1})\end{aligned}$$

In (D.2) , we can find a unique solution $\mathrm{x}_1^{*} \in (0,1), then$

$$\begin{cases} \mathrm{V}_{\mathrm{b}}^{*} = \dfrac{(1-\kappa)Pc}{r} \cdot \dfrac{\gamma + \beta(x_1^{*})^{\beta+\gamma} - (\beta+\gamma)(x_1^{*})^{\beta}}{(\gamma+1) + (\beta-1)(x_1^{*})^{\beta+\gamma} - \lambda(\beta+\gamma)(x_1^{*})^{\beta-1}} \\ \mathrm{V}_{\mathrm{con}}^{*} = \dfrac{Pc}{r} \cdot \dfrac{\beta + \gamma(x_1^{*})^{\beta+\gamma} - (\beta+\gamma)(x_1^{*})^{\gamma}}{(\lambda(\beta-1) + \lambda(\gamma+1)(x_1^{*})^{\beta+\gamma} - (1-\rho)(\beta+\gamma)(x_1^{*})^{\gamma+1})} \end{cases}$$

Lemma D.2

Given V_b^* and V_{con}^* solved from (4.2) and (4.3), consider an equation

(D.3) $E_1(V; V_b^*, V_{con}^*) = (1\text{-}\lambda)V$

Equation(D.3) has at most a unique solution in the interval (V_b^*, V_{con}^*)

Denote it to be k_1 if such solution exists or set $k_1 = V_{con}^*$ otherwise.

Let $K_1 = \lambda k_1$, we have

$$E_1^*(V; V_b^*, V_{con}^*) \geq h(V) \text{ and } D_1^*(V; V_b^*, V_{con}^*) \geq \lambda V$$

for any $V \in [V_b^*, V_{con}^*]$

NUMERICAL REAULTS

r	0.08	c	0.07
κ	0.35	λ	0.2
δ	0.06	P	100
σ	0.22	K	100
ρ	0.50		
β	0.012526784	γ	0.245073447
X1*	0.039830749		
Vb*	36.43	Vcal*	914.62
p	0.012526784	q	0.245073447
k1	437.55	K1	87.51

NUMERICAL REAULTS

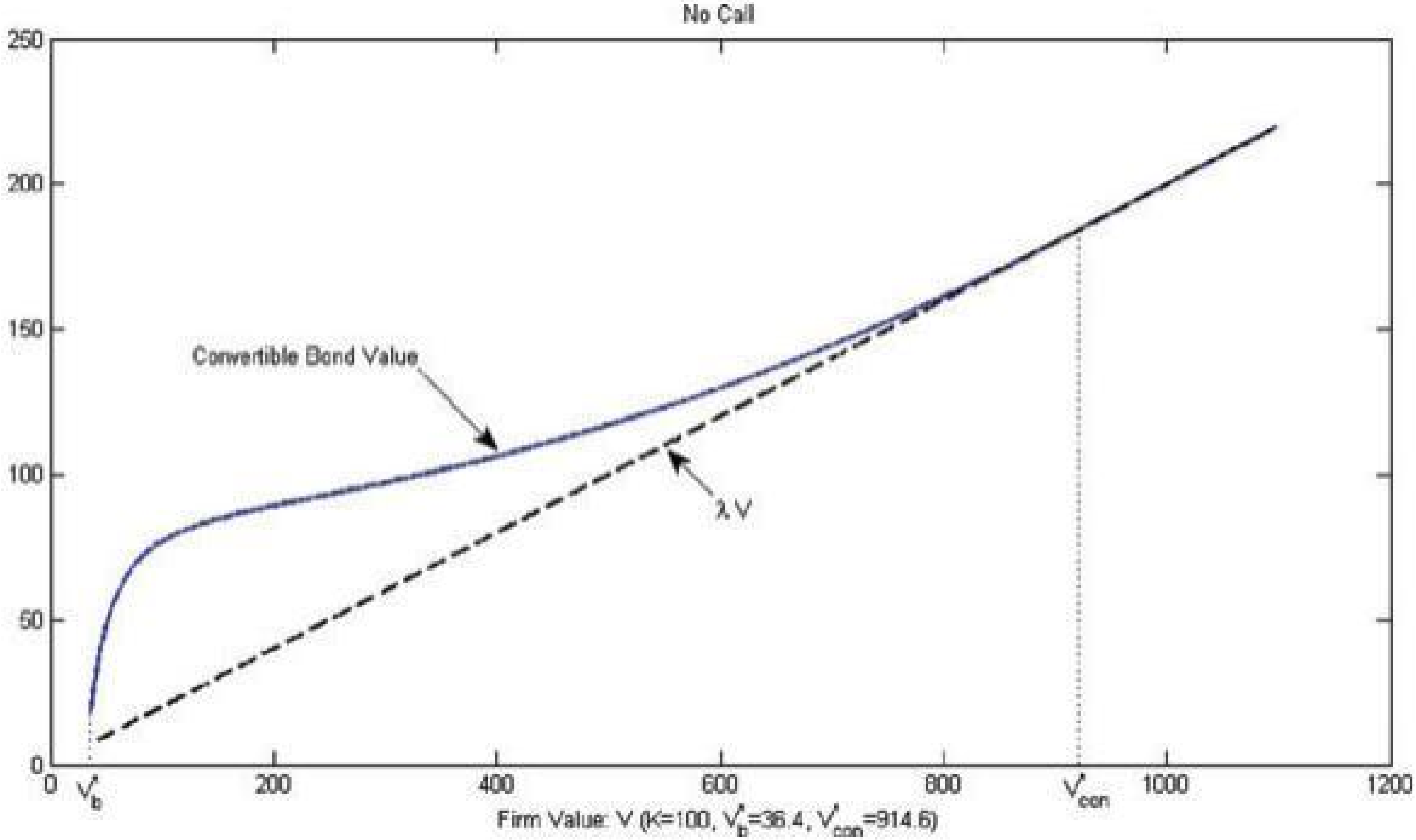

FIGURE 5.1. The convertible bond value in a case with a high call price. The default barrier $V_b^* = 36.43$ and the conversion barrier $V_{con}^* = 914.62$. The shareholder will never call the debt voluntarily.

Early and Late Calls

Theorem 4.2

When $K \le K_1$, a Nash equilibrium to the game (2.6 and 2.7) is formed if the bondholder converts her security at the moment

$$\tau^*_{\text{con}} = \inf\{t \ge 0 : V_t \ge V^*_{con}\}$$

and the shareholder declares bankruptcy and call at

$$\tau^*_{\text{b}} = \inf\{t \ge 0 : V_t \ge V^*_b\}$$

and $\tau^*_{\text{cal}} = \inf\{t \ge 0 : V_t \in [V^*_{cal,1}, V^*_{cal,2}]\}$, *respectively.*

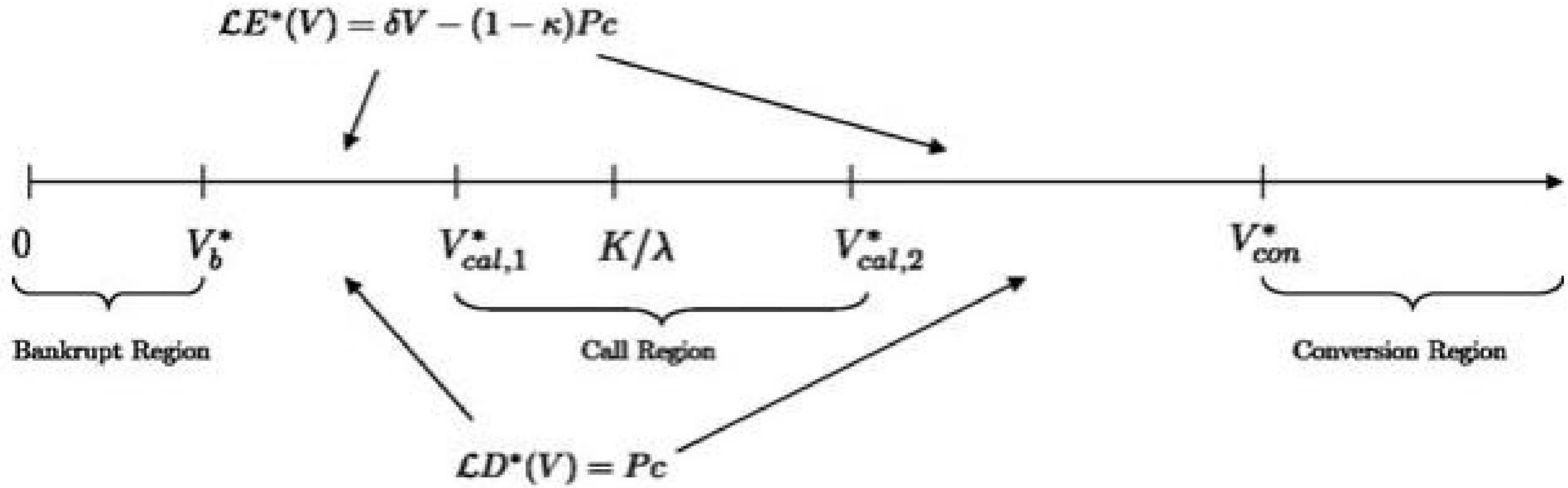

FIGURE 4.2. The optimal conversion and bankruptcy regions when $K > K_1$. The shareholder declares bankruptcy on $\mathcal{S}_E \cap [0, K] = [0, V_b^*]$ and makes a call on $\mathcal{S}_E \cap (K, +\infty) = [V^*_{cal,1}, V^*_{cal,2}]$. The bondholder converts on $\mathcal{S}_D = [V^*_{con}, +\infty)$.

On this equilibrium

$$
(\mathrm{E}^*(\mathrm{v}), \mathrm{D}^*(\mathrm{v})) = \begin{cases}
(0, (1-\rho)\mathrm{V}) & \text{if } V \le V_b^* \\
(\mathrm{E}_2(\mathrm{V}; \mathrm{V}_\mathrm{b}^*, \mathrm{V}_{\mathrm{cal},1}^*), \mathrm{D}_2(\mathrm{V}; \mathrm{V}_\mathrm{b}^*, \mathrm{V}_{\mathrm{cal},1}^*)) & \text{if } V_b^* < V \le V_{cal,1}^* \\
(V - K, K) & \text{if } V_{cal,1}^* < V \le K/\lambda \\
((1-\lambda)V, \lambda V) & \text{if } K/\lambda < V \le V_{cal,2}^* \\
(\mathrm{E}_3(\mathrm{V}; \mathrm{V}_\mathrm{b}^*, \mathrm{V}_{\mathrm{cal},1}^*), \mathrm{D}_3(\mathrm{V}; \mathrm{V}_\mathrm{b}^*, \mathrm{V}_{\mathrm{cal},1}^*)) & \text{if } V_{cal,2}^* < V \le V_{cal}^* \\
((1-\lambda)V, \lambda V) & \text{if } V_{cal}^* < V
\end{cases}
$$

where

$$D_2(V;V_b^*,V_{cal,1}^*) = V - \frac{(1-\kappa)Pc}{r} + (\frac{(1-\kappa)Pc}{r} - V_b^*)p(V;V_b^*,V_{cal,1}^*)$$

$$+(\frac{(1-\kappa)Pc}{r} - K)q(V;V_b^*,V_{cal,1}^*)$$

$$E_2(V;V_b^*,V_{cal,1}^*) = \frac{Pc}{r} + ((1-\rho)V_b^* - \frac{Pc}{r})p(V;V_b^*,V_{cal,1}^*) + (K - \frac{Pc}{r})q(V;V_b^*,V_{cal,1}^*)$$

$$D_3(V;V_{cal,2}^*,V_{con}^*) = V - \frac{(1-\kappa)Pc}{r} + (\frac{(1-\kappa)Pc}{r} - \lambda V_{cal,2}^*)p(V;V_{cal,2}^*,V_{con}^*)$$

$$+(\frac{(1-\kappa)Pc}{r} - \lambda V_{con}^*)q(V;V_{cal,2}^*,V_{con}^*)$$

$$E_2(V;V_{cal,2}^*,V_{con}^*) = \frac{Pc}{r} + (\lambda V_{cal,2}^* - \frac{Pc}{r})p(V;V_{cal,2}^*,V_{con}^*) + (\lambda V_{con}^* - \frac{Pc}{r})q(V;V_{cal,2}^*,V_{con}^*)$$

Early and Late Calls

the ODEs :

$\mathrm{LD}^*(\mathrm{V}) = \mathrm{Pc}$ and $\mathrm{LE}^*(\mathrm{V}) = \delta\mathrm{V} - (1-\kappa)\mathrm{Pc}$

Take the interval $(\mathrm{V}_{\mathrm{b}}^*, \mathrm{V}_{\mathrm{cal,1}}^*)$, for instance.As $\mathrm{D}^*(\mathrm{V}) = (1-\rho)\mathrm{V}, \mathrm{E}^*(\mathrm{V}) = 0$ for $\mathrm{V} \le \mathrm{V}_{\mathrm{b}}^*$ and $\mathrm{D}^*(\mathrm{V}) = K$, $\mathrm{E}^*(\mathrm{V}) = V - K$ for $\mathrm{V} \in [\mathrm{V}_{\mathrm{cal,1}}^*, K/\lambda]$, the continuous property of D^* and E^* requires that

$$\begin{cases} \mathrm{D}^*(\mathrm{V}_{\mathrm{b}}^*) = (1-\rho)\mathrm{V}_{\mathrm{b}}^* \\ \mathrm{E}^*(\mathrm{V}_{\mathrm{b}}^*) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{D}^*(\mathrm{V}_{\mathrm{cal,1}}^*) = K \\ \mathrm{E}^*(\mathrm{V}_{\mathrm{cal,1}}^*) = \mathrm{V}_{\mathrm{cal,1}}^* - K \end{cases}$$

We can easy show that $D^*(V) = D_2(V; V_b^*, V_{cal,1}^*)$ and $E^*(V) = E_2(V; V_b^*, V_{cal,1}^*)$ are the solutions.Similar to D^* and E^* in $(V_{cal,2}^*, V_{con}^*)$

When $\mathrm{K} < \mathrm{K}_2$ or $\mathrm{K} < \mathrm{K}_3$, invoke the smooth requirement in the variational inequality system once again to find the qptimal boundaries $V_b^*, V_{cal,1}^*, V_{cal,2}^*$ and V_{con}^*.According to Condttions 4 and 5 in the variational inequality system, E^* should smooth at $\mathrm{V} = V_b^*$ and $\mathrm{V} = V_{cal,1}^*$, which requires :

$$\frac{\mathrm{dE}_3^*(V; V_b^*, V_{cal,1}^*)}{\mathrm{dV}} \Big|_{V = V_b^* +} = 0 \text{ and } \frac{\mathrm{dE}_3^*(V; V_b^*, V_{cal,1}^*)}{\mathrm{dV}} \Big|_{V = V_{cal,1-}^*} = 1 \cdots (4.4)$$

Early and Late Calls

LemmaD.3(i)

Similarly, when $K < K_3$, the following two equations can be used to determine $V^*_{cal,2}$ and V^*_{con} :

$$\frac{dE_3^*(V;V^*_{cal,2},V^*_{con})}{dV}\Big|_{V=V^*_{cal.2}+} = 1-\lambda \text{ and } \frac{dE_3^*(V;V^*_{cal,2},V^*_{con})}{dV}\Big|_{V=V^*_{con}-} = \lambda$$

LemmaD.3(ii) shows the existence and uniqueness of the solutions.

By Theorem 4.2, if $K_2 \le K \le K_1, V^*_{cal,1} = K/\lambda$. In this case, we use

$$\frac{dE_2^*(V;V_b^*,K/\lambda)}{dV}\Big|_{V=V_b^*+} = 0$$

to solve for the optimal bankrupt boundary V_b^*. When $K_3 \le K \le K_1, V^*_{cal,2} = K/\lambda$ and the following

$$\frac{dD_2^*(V;K/\lambda,V^*_{con})}{dV}\Big|_{V=V^*_{con}-} = \lambda$$

is used to determined the conversion boundary V^*_{con}.

NUMERICAL REAULTS

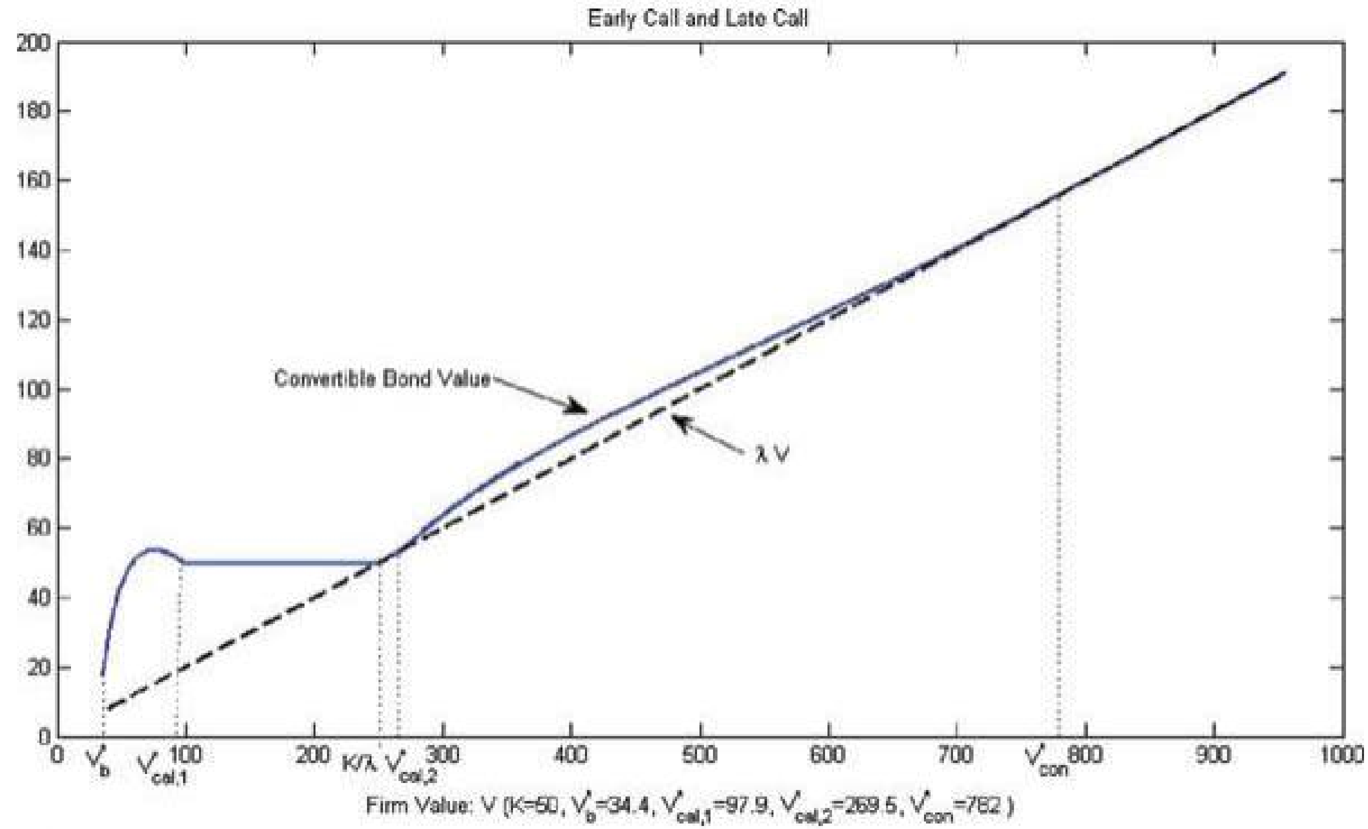

FIGURE 5.2. The convertible bond value in a case with a lower call price. $V_b^* = 35.44$, $V_{cal,1}^* = 97.90$, $V_{cal,2}^* = 269.51$, and $V_{con}^* = 782.00$. Once the initial company value falls in the interval $(V_{cal,1}^*, K/\lambda)$, the shareholder will call the debt immediately to force the bondholder to surrender her security. Hence, the bond value equals $K = 50$ in this interval, which is reflected by the horizontal straight line between $V_{cal,1}^*$ and K/λ. In the interval $(K/\lambda, V_{cal,2})$, the bondholder chooses to convert in response to the call from the shareholder. Hence, the bond value and its conversion value λV coincide.

Comparative Static

TABLE 5.2
Effects of Various Parameters on the Optimal Strategies

	V_b^*	$V_{cal,1}^*$	$V_{cal,2}^*$	V_{con}^*
r				
0.07	36.49	80.49	271.20	795.59
0.08	35.44	97.90	269.51	782.00
0.09	33.98	250.00	268.14	768.97
c				
0.06	30.96	250.00	250.00	656.96
0.07	35.44	97.90	269.51	782.00
0.08	38.49	74.11	308.01	893.72
δ				
0.05	36.61	92.71	318.83	928.80
0.06	35.44	97.90	269.51	782.00
0.07	34.17	104.33	250.00	664.61
κ				
0.15	40.84	65.28	423.86	675.69
0.25	38.68	73.22	339.04	733.19
0.35	35.44	97.90	269.51	782.00
0.45	30.58	250.00	250.00	795.60

Note: The defaulting parameter used is $K = 50$. We vary the parameter of interest each time and keep all other parameters the same as the base case in Table 5.1.

Comparative Static

TABLE 5.3
Effects of Various Parameters on the Convertible Bond Value

Interest rate r	0.06	0.07	0.08	0.09
Bond value	105.60	105.25	104.88	104.49
Coupon rate c	0.06	0.07	0.08	0.09
Bond value	101.88	104.88	107.12	108.57
Paying out rate δ	0.05	0.06	0.07	0.08
Bond value	106.62	104.88	102.39	100.45
Tax rate κ	0.15	0.25	0.35	0.45
Bond value	100.67	102.42	104.88	105.74

Note: The defaulting parameters used are $K = 50$ and $V_0 = 500$. We vary r, c, δ, and κ in each row and keep all other parameters the same as those in Table 5.1.

Negative and positive Equity Returns

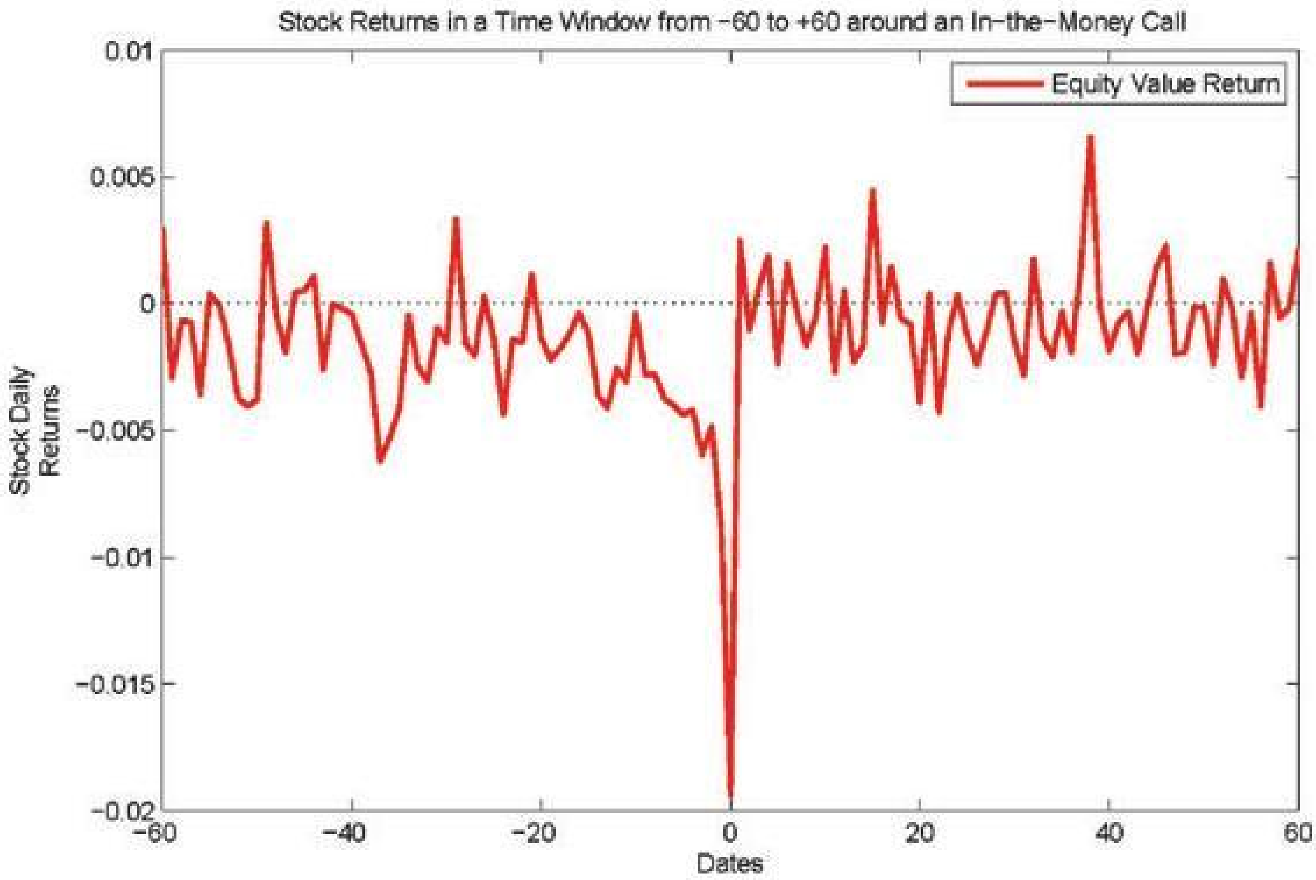

FIGURE 5.3. Daily returns of equity in a time window from 60 days before the call to 60 days after. We use the default parameters in Table 5.1 and let $V_0 = 300$. $K = 50$. The call boundary is $V^*_{cal,2} = 269.51$ and the conversion boundary is $V^*_{con} = 782$. We simulate 100 sample paths in which a call happens and plot the average.

Negative and positive Equity Returns

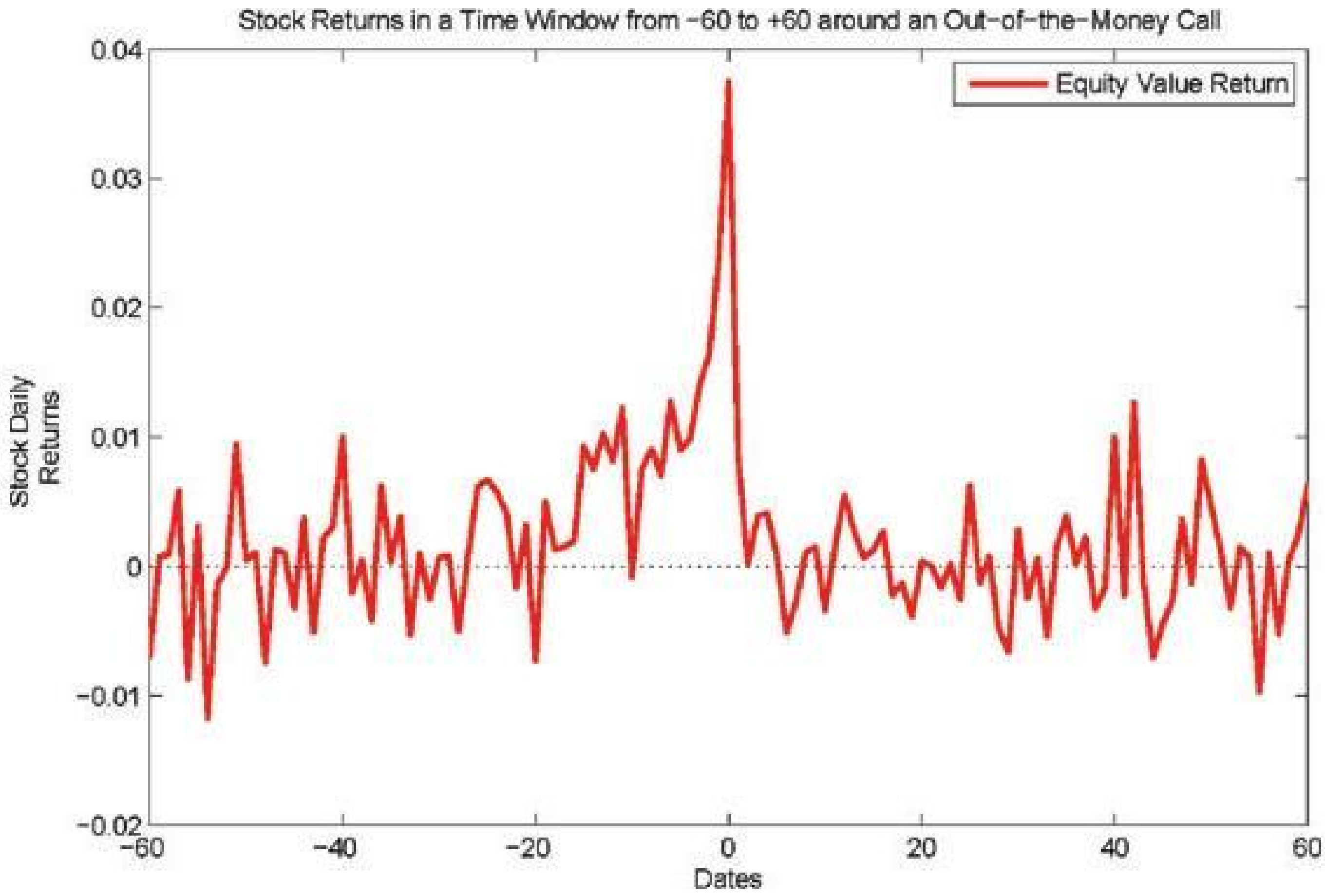

FIGURE 5.4. Daily returns of equity in a time window from 60 days before the call to 60 days after. We use the default parameters in Table 5.1 and let $V_0 = 90$. The call price is 50. The call boundary is $V^*_{cal,1} = 97.90$ and the default boundary is $V^*_b = 35.44$. We simulate 100 sample paths and draw the average.